240Algebra1-141006 Hour 13: Linear combinations, linear independence

September-18-14 6:29 PM

HW4 on web by midnight.
HW3 due <u>Tue</u> Oct 7 @ tuto.
Read Along sections 1.5–1.7.
Riddle Along ?

Today: linear combinations, lin. independence.

Reminder. $S \subset V$; $\text{span}(S) = \{u : u \text{ is a l.c. of vectors in } S\}$

$$\Leftrightarrow \{u : \exists \alpha_i \in F,\ u_i \in S \text{ s.t. } u = \sum \alpha_i u_i\}$$

always a subspace!

$S \subset V$ "generates" or "spans" $V \Leftrightarrow \text{span } S = V$

<u>Examples</u> In $V = M_{2\times 2}(\mathbb{R})$ $M_1 = \begin{pmatrix}1&0\\0&0\end{pmatrix}$ $M_2 = \begin{pmatrix}0&1\\0&0\end{pmatrix}$ $M_3 = \begin{pmatrix}0&0\\1&0\end{pmatrix}$ start line

$M_4 = \begin{pmatrix}0&0\\0&1\end{pmatrix}$ $N_1 = \begin{pmatrix}0&1\\1&1\end{pmatrix}$ ... $N_4 = \begin{pmatrix}1&1\\1&0\end{pmatrix}$, Then

$M_1 \dots M_4$ & $N_1 \dots N_4$ generate V, but

$M_1 \dots M_3$ & $N_1 \dots N_3$ do not.

$\left(\underset{\begin{pmatrix}0&1\\1&1\end{pmatrix}}{N_1}, \underset{\begin{pmatrix}1&0\\1&1\end{pmatrix}}{N_2}, \underset{\begin{pmatrix}1&1\\0&1\end{pmatrix}}{N_3} \text{ are in } \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix} : a+b+c = 2d\right\} \right)$

Aside: If $S_1 \subset W$ then $\text{span}(S_1) \subset W$

So if $S_1 \subset \text{span}(S_2)$ then $\text{span}(S_1) \subset \text{span}(S_2)$

post mortem: I should have written $M_1 = \frac{1}{3}(N_1 + N_2 + N_3 + N_4 - 3N_1)$ first, then "a l.c. is a l.c.", then done the aside backwards. As it was, too many easy steps made the overall figure appear harder than it was.

<u>Def</u> A subset $S \subset V$ is "lin. dep" if it is "wasteful".

I.e., If $\exists\, a_i \in F$ not all 0 & $u_i \in S$ s.t. $\sum a_i u_i = 0$.

Otherwise, it is "lin. indep."

<u>Examples</u> $\{e_i\}$ ✓, $\cancel{\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}}$

done line

<u>Comments</u> 1. $\emptyset$ is lin. indep.

2. $\{u\}$ is lin indep iff $u \neq 0$.

3. Suppose $S_1 \subset S_2 \subset V$. Then

   a. If S_1 is dep, so is S_2

   b. If S_2 is indep, so is S_1

4. If S is lin indep in V and $v \in V$, then $S \cup \{v\}$ is lin. dep. iff $v \in \text{span}(S)$.

<u>Def</u> Basis $\beta \subset V$

<u>Thm</u> A subset $\beta \subset V$ is a basis iff every $v \in V$ can be expressed in a unique way as a l.c. of elements of β

<u>Thm</u> If a finite set S generates a V.S. V, then there is a subset $\beta \subseteq S$ which is a basis of V

<u>Pf</u> Let β be a lin indep subset of S which is of maximal size. Then every $v \in S \setminus \beta$ satisfies $v \in \text{span}\,\beta$, so $S \subset \text{span}\,\beta$, so $\text{span}\,S \subset \text{span}\,\beta$.