240AlgebraI-140911, Hours 2-3: The Notion of a "Field"

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Today's Menu: The very basic of "fields".

Read Along. Appendices A–D.

Riddle Along. $1 = \sqrt{1} = \sqrt{(-1)\cdot(-1)} = \sqrt{-1}\cdot\sqrt{-1} = i\cdot i = -1$

on board

The real numbers: a set $\mathbb{R}$ w/ two binary ops $+$ & $\times$ and two special elements 0 & 1 s.t.

R1 $a+b=b+a$ $ab=ba$

R2 Assoc.

R3 0, 1

R4 negatives & inverses

R5 Distributivity.

Much of algebra, though not all, follows:

Follows: $(a+b)(a-b)=a^2-b^2$

Doesn't follow: $\forall a\ \exists x$ s.t. $a=x^2$ or $a=-x^2$

A Field: $(F,+,\times,0\neq 1)$ s.t.

F1 Commutativity:

F2 Associativity:

F3 Units

F4 Inverses

F5 Distributivity

Examples 1. The reals $\mathbb{R}$.

2. The rationals $\mathbb{Q}$

3. The complex numbers $\mathbb{C}=\{a+b\sqrt{-1}\}$

4. 0, 1 with

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

($\mathbb{Z}/2$)

5. 0, 1, 2, 3, 4, 5, 6 with a funny def. of $+$, $\times$. ($\mathbb{Z}/7$)

6. $\mathbb{Z}/p$ for any prime p.

Proofs...

Thm 1. $a+b=c+b \Rightarrow a=c$

2. $a\cdot b=c\cdot b$, $b\neq 0 \Rightarrow a=c$

3. If $0'$ is like 0, then $0'=0$

4 If $1'$ is like 1, then $1'=1$

5 If $a+b=0=a+b'$ then $b=b'$ (so we can define $-a$)

6. If $a\neq 1$ & $ab=1=ab' \Rightarrow b=b'$ (so we can define a^{-1})

done line

If: subtraction $a-b$,

6. If $a \neq 1$ & $ab = 1 = ab \Rightarrow b = b$
(so we can define a^{-1})

7. $-(-a) = a$, $(a^{-1})^{-1} = a$

Def: subtraction $a - b$, division a/b when $b \neq 0$

8. $a \cdot 0 = 0$

9. There's no 0^{-1}

That's hard!!

10. $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$

11. $(-a)(-b) = a \cdot b$