Wednesday-1 AKT on 140108 - Scratch: Maxwell, pushforwards

January-08-14 9:26 AM

6. The relation w/ magnetic Fields.

Maxwell's equations

I. $\nabla \cdot E = \frac{\rho}{\epsilon_0}$

II. $\nabla \times E = -\frac{\partial B}{\partial t}$

III. $\nabla \cdot B = 0$

IV. $c^2 \nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t}$

[Conservation of charge

$\nabla \cdot j = -\frac{\partial \rho}{\partial t}$

$\xrightarrow[\epsilon_0 = c = 1]{\text{statics}}$

div $B = 0$ "no magnetic monopoles"

curl $B = j$ "electromagnets"

div $j = 0$ "conservation of charge"

(also div $E = \rho$
curl $E = 0$)

$\Longleftrightarrow$ $B \in \Omega^1(\mathbb{R}^3)$, $j \in \Omega^2(\mathbb{R}^3)$

$d{*}B = 0$

$dB = j$

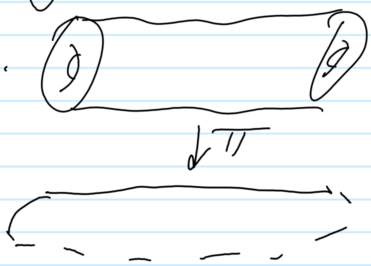
$dj = 0$

Pushforwards / integration along the fibers:

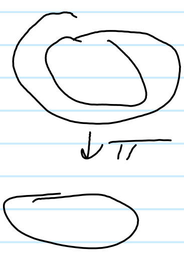
$$\begin{array}{ccc} F^k & \longrightarrow & T \\ & & \downarrow \pi \\ & & M \end{array}$$

All are oriented mflds & smooth maps, perhaps w/ bndry

Examples: 1.



2. $I \longrightarrow$



<u>Claim/def</u> There is a unique

$\pi_*: \Omega^n(T) \to \Omega^{n-k}(M)$

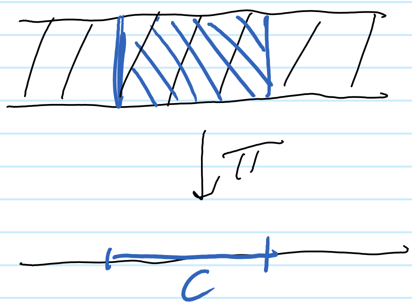
s.t. for every $w \in \Omega^n(T)$ & every $(n-k)$-cycle C in M,

$$\int_C \pi_*(w) = \int_{\pi^{-1}(C)} w$$

Thm ("Stokes' for pushforwards")

with $\begin{tikzcd} \partial F \arrow[r] & T \arrow[d, "\partial\pi"] \\ & M \end{tikzcd}$, $\qquad d\pi_* \omega = \pi_* d\omega - (\partial\pi)_* \omega$

PF



$\omega \in \Omega^n(T)$

c a $(n-k+1)$-cycle

$d\theta = 0$!

$$d\,\pi_{x*}\left(\phi^*(\omega) \wedge \pi_y^*\theta\right) = \pi_{x*}\left(d(\phi^*\omega \wedge \pi_y^*\theta)\right)$$
$$- (\partial\pi_x)_*\left[\phi^*\omega \wedge \pi_y^*\theta\right]$$

$$= -\theta$$

From Bott & Tu

Proposition 6.15 (Projection Formula). (a) *Let $\pi: E \to M$ be an oriented rank n vector bundle, τ a form on M and ω a form on E with compact support along the fiber. Then*

$$\pi_*((\pi^*\tau)\cdot\omega) = \tau \cdot \pi_*\omega.$$

(b) *Suppose in addition that M is oriented of dimension m, $\omega \in \Omega^q_{cv}(E)$,* and $\tau \in \Omega_c^{m+n-q}(M)$. *Then with the local product orientation on E*

$$\int_E (\pi^*\tau)\wedge\omega = \int_M \tau \wedge \pi_*\omega.$$