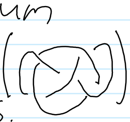
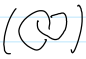
Wednesday-1 AKT on 140108: The Gauss Linking Number combinatorially and as an integral

January-05-14 5:24 PM

Scheduling!

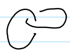
1. The linking number as a sum
   a. Definition, $\ell(\bigcirc\bigcirc)=\ell($  $)\neq\ell($  $)$
   b. Invariance under R moves.
   c. Invariance under "homotopy"

2. The linking number as an integral.

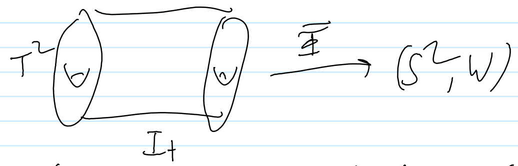
Given $\gamma_1,\gamma_2: S^1\to\mathbb{R}^3$ w/ disjoint images,

$$\Phi: T^2=S^1\times S^1\longrightarrow S^2 \quad\text{by}\quad s_1,s_2\to\frac{\gamma_1(s_1)-\gamma_2(s_2)}{\|\quad\|}$$

$$\ell(\gamma_1,\gamma_2)=\int_{T^2}\Phi^*w,$$ where $w\in\Omega^2(S^2)$ is the unit volume form.

a. Compute for $\bigcirc\bigcirc$ & for . (assuming invariance)

b. Proof of invariance: $\gamma_1^t,\gamma_2^t,\bar\Phi$



$$0=\int\bar\Phi^*dw=\int d\bar\Phi^*w=\int_{T_0^2}\Phi_0^*w-\int_{T_1^2}\Phi_1^*w$$

c. Can replace w by any other w' with $\int_{S^2}w'=1$

$\Rightarrow 2^{nd}$ proof of $\ell^{comb}=\ell^{int}$

d. Explicit Formula: $w=\frac{1}{4\pi}(x\,dy\wedge dz+y\,dz\wedge dx+z\,dx\wedge dy)$

$$=\frac{1}{8\pi}\epsilon_{ijk}x^i dx^j\wedge dx^k$$

(PF w is invariant, & $\int_{B^3}dw=\frac{3}{4\pi}Vol(B^3)=1$)

So
$$\ell(\gamma_1,\gamma_2)=\frac{1}{4\pi}\int ds_1ds_2\,\frac{\dot\gamma_1^i(s_1)\,\dot\gamma_2^j(s_2)\,(\gamma_1^k-\gamma_2^k)}{\|\gamma_1-\gamma_2\|^3}\epsilon_{ijk}$$

done

Thu

3 The linking number as a degree: Given $\Phi: M^n\to N^n$ between oriented manifolds,

$$\deg\Phi=\frac{\Phi_*[M]}{\Phi_*[N]}=\int_M\Phi^*w_N=\sum_{x\in\Phi^{-1}(y)}\deg_x\Phi$$

is a homotopy invariant. $\ell(\gamma_1,\gamma_2)=\deg\Phi^{\gamma_1,\gamma_2}$

is a homotopy invariant. $\ell(\gamma_1,\gamma_2) = \deg \Phi^{\gamma_1,\gamma_2}$

4. <u>Claim</u>

$\ell(\gamma_1,\gamma_2) = $ "$\langle \gamma_1 | \mathrm{curl}^{-1} | \gamma_2 \rangle$"

<u>notes</u> 0. should be smoothed out...
1. Makes intuitive sense!
2. Makes sense only if γ_1 & γ_2 are closed.

3. related to Maxwell's eq'ns.

Better w/ diff. forms ... recall

$$\Omega^0\mathbb{R}^3 \xrightarrow{d} \Omega^1\mathbb{R}^3 \xrightarrow{d} \Omega^2\mathbb{R}^3 \xrightarrow{d} \Omega^3\mathbb{R}^3$$

$$\text{fun} \xrightarrow{\text{grad}} \text{v.f.} \xrightarrow{\text{curl}} \text{v.f.} \xrightarrow{\text{div}} \text{fun}$$

5. <u>claim</u> $\ell(\gamma_1,\gamma_2) = $ "$\langle \gamma_1 | d^{-1} | \gamma_2 \rangle$"

a. In $\mathbb{R}^3$, curves $\Leftrightarrow$ currents $\Leftrightarrow$ 2-forms
closed curves $\Leftrightarrow$ closed 2-forms
$\gamma \Leftrightarrow \theta_\gamma$

$$\int_\gamma \lambda = \int_{\mathbb{R}^3} \lambda \wedge \theta_\gamma$$

b. ℓ: $\gamma_2 \longrightarrow \theta_{\gamma_2} \longrightarrow$ some σ_{γ_2} s.t. $d\sigma_{\gamma_2} = \theta_{\gamma_2}$

$$\longrightarrow \int_{\gamma_1} \sigma_{\gamma_2} = \text{“} \int_{\gamma_1} d^{-1}\theta_{\gamma_2} \text{”}$$

(it doesn't matter which σ you take)!

..... but how do we find d^{-1}?

<u>Claim</u> If $\theta \in \Omega^2(\mathbb{R}^3)$ & $d\theta = 0$, consider

$\phi : \mathbb{R}^3_x \times \mathbb{R}^3_y \longrightarrow S^2 \quad (x,y) \longmapsto \frac{x-y}{\|\ \ \|}$ $\quad$ ω vol. on S^2

$\pi_{x,y} : \mathbb{R}^3_x \times \mathbb{R}^3_y \longrightarrow \mathbb{R}^3_x, \mathbb{R}^3_y$ projections

and set

$$\sigma = \int_{\mathbb{R}^3_y} \left(\phi^*(\omega) \wedge \pi_y^*\theta\right) \in \Omega^1(\mathbb{R}^3_x)$$

Then $d\sigma = \theta$ $\quad$ "a formula for d^{-1}"

<u>claim</u> Indeed, $\int_{\gamma_1} d^{-1}\theta_{\gamma_2}$ is $\ell(\gamma_1,\gamma_2)$