Solution of Homework Assignment 5

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The assignment is at http://drorbn.net/index.php?title=AKT-14/Homework_Assignment_5

Just one question.

Prove

$$\dim \mathcal{A}_n = \dim \mathcal{A}_n^r + \dim \mathcal{A}_{n-1}$$

Proof. Let $\hat{\theta}: \mathcal{A} \to \mathcal{A}$ be the operator "multiplication by the chord diagram $\theta = \ominus$".

We have a short sequence

$$0 \to \mathcal{A}_{n-1} \xrightarrow{\hat{\theta}} \mathcal{A}_n \xrightarrow{\pi} \mathcal{A}_n^r \to 0,$$

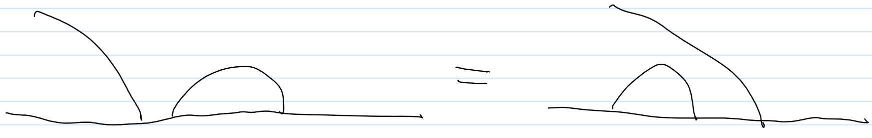
where π is the projection of a space to a quotient thereof. The problem follows from the exactness of this sequence, and we claim that it is indeed exact.

Exactness at $\mathcal{A}_n^r$ is the surjectivity of π, which is obvious.

Exactness at $\mathcal{A}_n$ is

$\ker \pi \supset \operatorname{im} \hat{\theta}$: obvious.

$\ker \pi \subset \operatorname{im} \hat{\theta}$: Note that



So "short" chord can slide to the left.

Hence every diagram that has a short chord is in the image of $\hat{\theta}$, hence every FI relation is in the image of $\hat{\theta}$.

Exactness at $\mathcal{A}_{n-1}$, the injectivity of $\hat{\theta}$, is the hardest part of the proof. Let

$$\frac{d}{d\theta} : A_n \longrightarrow A_{n-1}$$

be "the sum of the n ways of dropping one chord from a diagram in A_n to get a diagram in A_{n-1}".

Fact 1 $\frac{d}{d\theta}$ is well defined mod 4T. $\square$

Fact 2 $\left[\frac{d}{d\theta}, \hat{\theta}\right] := \frac{d}{d\theta} \circ \hat{\theta} - \hat{\theta} \circ \frac{d}{d\theta} = I$ $\square$

Fact 3 $\left[\left(\frac{d}{d\theta}\right)^k, \hat{\theta}\right] = k\left(\frac{d}{d\theta}\right)^{k-1}$ $\square$

Note that $\frac{d}{d\theta}$ is nilpotent - $\left.\left(\frac{d}{d\theta}\right)^{n+1}\right|_{A_n} = 0$, and

set

$$\psi := \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \hat{\theta}^k \circ \left(\frac{d}{d\theta}\right)^{k+1}$$

I claim that $\psi \circ \hat{\theta} = I$, and hence $\hat{\theta}$ is injective. Indeed,

$$\psi \circ \hat{\theta} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \hat{\theta}^k \left(\frac{d}{d\theta}\right)^{k+1} \hat{\theta} =$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \hat{\theta}^k \hat{\theta} \left(\frac{d}{d\theta}\right)^{k+1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \hat{\theta}^k \left[\left(\frac{d}{d\theta}\right)^{k+1}, \hat{\theta}\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \hat{\theta}^{k+1} \left(\frac{d}{d\theta}\right)^{k+1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \hat{\theta}^k \left(\frac{d}{d\theta}\right)^k$$

$$= -\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \hat{\theta}^k \left(\frac{d}{d\theta}\right)^k + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \hat{\theta}^k \left(\frac{d}{d\theta}\right)^k$$

$$= I. \qquad \square$$