Friday-4 AKT on 140131: The linking number from Chern-Simons

January-28-14 9:15 AM

on board

$\Lambda: V\to V^*$, symmetric & P.D., $\varphi_1, \varphi_2 \in V^*$, $Z := \int_{x\in V} dx\, e^{-\frac{1}{2} v\cdot\Lambda v}$,

Then $Z^{-1}\int_{x\in V} dx\, e^{-\frac{1}{2} v\cdot\Lambda v}\varphi_1(v)\varphi_2(v) = \varphi_1\Lambda^{-1}\varphi_2$

$$Z(\gamma_1,\gamma_2) := Z^{-1}\int_{A\in\Omega^1(\mathbb{R}^3)} \mathcal{D}A\, e^{\frac{i}{4\pi}\int_{\mathbb{R}^3} A\wedge dA}\int_{\gamma_1}A\cdot\int_{\gamma_2}A = C\langle\gamma_1|d^{-1}|\gamma_2\rangle$$

More precisely, $\Lambda: \{1\text{-forms}\} \to \langle\text{currents}\rangle$ j

$\{1\text{-forms}\} \xrightarrow{d} \{2\text{-forms}\}$, $\langle\text{currents}\rangle = \{2\text{-forms}\}$, $\int_D \alpha = \int_D j\cdot\vec{n}$

$\Lambda^{-1} \sim d^{-1} : \langle\text{currents}\} \to \Omega^1$

$j \to$ the 1-form λ s.t.

$$\int_{\partial D}\lambda = \int_D d\lambda = \int_D j\cdot\vec{n}$$

$\Lambda^{-1}\gamma_2$: The 1-form λ whose integral around a small loop ∂D is 1 iff γ_2 pierces D positively

$$\langle\gamma_1|\Lambda^{-1}|\gamma_2\rangle = l(\gamma_1,\gamma_2) \ !$$

Claim If $\alpha\in\Omega^2(\mathbb{R}^3)$ & $d\alpha=0$, consider

$\phi: \mathbb{R}^3_x\times\mathbb{R}^3_y \to S^2$ $(x,y)\mapsto \frac{x-y}{\|x-y\|}$ w vol. on S^2

$\pi_{x,y}: \mathbb{R}^3_x\times\mathbb{R}^3_y \to \mathbb{R}^3_x, \mathbb{R}^3_y$ projections

and set

$$\sigma = \int_{\mathbb{R}^3_y}\left(\phi^*(w)\wedge\pi_y^*\alpha\right) \in \Omega^1(\mathbb{R}^3_x)$$

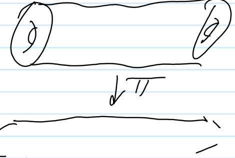
"a formula for d^{-1}"

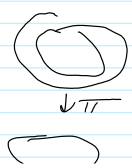
Then $d\sigma = \alpha$

Pushforwards / integration along the fibers: long aside

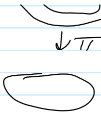
$$\begin{array}{ccc} F^k & \longrightarrow & T \\ & & \downarrow \pi \\ & & M \end{array}$$

All are oriented mflds & smooth maps, perhaps w/ bndry

Examples: 1. 

2. $I \longrightarrow$ 

Claim/def There is a unique



Claim/def There is a unique

$$\pi_*: \Omega^n(T) \to \Omega^{n-k}(M)$$

s.t. for every $w \in \Omega^n(T)$ & every $(n-k)$-cycle c in M,

$$\int_c \pi_*(w) = \int_{\pi^{-1}(c)} w$$

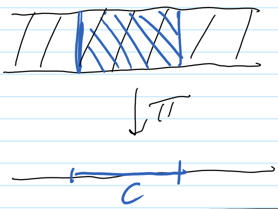
done line

Easy property:
$$\begin{array}{ccc} F^k \to & T & w \\ & \downarrow \pi & \\ & M & \alpha \end{array} \qquad \pi_*(w \wedge \pi^* \alpha) = (\pi_* w) \wedge \alpha$$

Thm ("Stokes' for pushforwards")

with
$$\begin{array}{cc} \partial F \to & T' \\ & \downarrow \partial\pi \\ & M \end{array}, \qquad d\pi_* w = \pi_* dw - (\partial \pi)_* w$$

Pf



$w \in \Omega^n(T)$

c a $(n-k+1)$-cycle

Proof of Claim
$$\begin{array}{ccccc} \mathbb{R}^3 \setminus pt \to & \tilde{C}_2(\mathbb{R}^3) & \xrightarrow{\phi} & S^2, w \\ & \downarrow \pi_x & \searrow \pi_y & \\ & \mathbb{R}^3_x & & \mathbb{R}^3_y, \alpha \end{array}$$

$\sigma = \pi_{x*}\left((\phi^* w) \wedge \pi_y^*(\alpha)\right)$

satisfies $-d\sigma = \alpha$, if $d\alpha = 0$

Pf $d\sigma = \pi_{x*}(0) - (\partial \pi_x)_*(\phi^* w \wedge \pi_y^* \alpha)$

$$= -(\partial \pi_x)_*(\phi^* w) \wedge \alpha = -\alpha$$