From Gaussian Integration to Feynman Diagrams

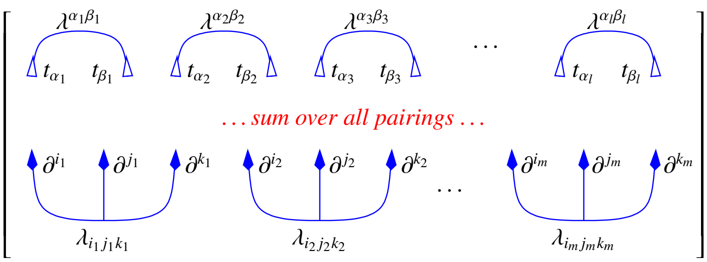
We wish to understand $\displaystyle\int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\,\mathrm{hol}_\gamma(A)\exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3}\mathrm{tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)\right].$

As a warm up, suppose (λ_{ij}) is a symmetric positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of $\mathbb{R}^n$, let $(t_i)_{i=1}^n$ be a set of "dual" variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

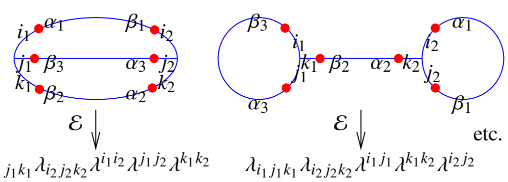
$$\int_{\mathbb{R}^n} e^{-\frac12\lambda_{ij}x^ix^j+\frac16\lambda_{ijk}x^ix^jx^k} = \int_{\mathbb{R}^n} e^{\frac16\lambda_{ijk}x^ix^jx^k}e^{-\frac12\lambda_{ij}x^ix^j}$$

$$= Ce^{\frac16\lambda_{ijk}\partial^i\partial^j\partial^k}e^{\frac12\lambda^{\alpha\beta}t_\alpha t_\beta}\Big|_{t_\alpha=0} = \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C}{6^m m!2^l l!}\left(\lambda_{ijk}\partial^i\partial^j\partial^k\right)^m\left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C}{6^m m!2^l l!}\left[\ \cdots\ \right]$$



$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C}{6^m m!2^l l!}\sum_{\substack{m\text{-vertex fully marked}\\ \text{Feynman diagrams } D}}\mathcal{E}(D)$$

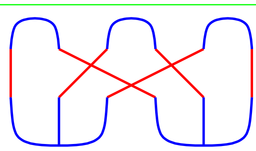
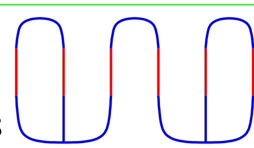


$$= C\sum_{\substack{\text{unmarked Feynman}\\ \text{diagrams } D}}\frac{\mathcal{E}(D)}{|\mathrm{Aut}(D)|}.$$

Claim. The number of pairings that produce a given unmarked Feynman diagram D is $\frac{6^m m!2^l l!}{|\mathrm{Aut}(D)|}$.

Proof of the Claim. The group $G_{m,l} := [(S_3)^m \rtimes S_m]\times[(S_2)^l\rtimes S_l]$ acts on the set of pairings, the action is transitive on the set of pairings P that produce a given D, and the stabilizer of any given P is $\mathrm{Aut}(D)$. □

Examples.

$|\mathrm{Aut}(D)| = 12$  $|\mathrm{Aut}(D)| = 8$ 

The Fourier Transform:

$(F: V\to\mathbb{C}) \Rightarrow (\tilde F: V^*\to\mathbb{C})$

via $\tilde F(\varphi) = \int_V F(v)e^{-i\langle\varphi,v\rangle}dv$.

Simple Facts:

1. $\tilde F(0) = \int_V F(v)dv$.
2. $\frac{\partial}{\partial\varphi_i}\tilde F \sim \widetilde{v^i F}$.
3. $\widetilde{\left(e^{Q/2}\right)} \sim e^{-Q^{-1}/2}$

   where $Q^{-1}(\varphi) := \langle\varphi, L^{-1}\varphi\rangle$

   (that's the heart of the Fourier Inversion Formula).

V: vector space

dV: Lebesgue's measure on V.

Q: A quadratic form on V; $Q(V) = \langle LV, V\rangle$ where $L: V\to V^*$ is linear

Compute $I = \int_V dv\, e^{\frac12 Q + P}$

$$\text{"="}\sum_{m=0}^\infty \frac{1}{m!}\int_V dv\, P^m e^{Q/2}$$

$$\sim \sum_{m=0}^\infty \frac{1}{m!}P^m(\partial_\varphi)e^{-\frac12 Q^{-1}(\varphi)}\Big|_{\varphi=0}$$

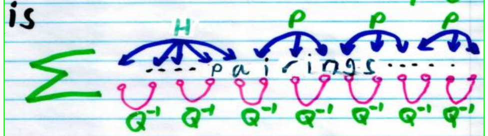
$\pi = 2 = i = 1$

$$= \sum_{m,n=0}^\infty \frac{(-1)^n}{2^n m! n!}P^m(\partial)(Q^{-1})^n\Big|_{\varphi=0}$$

So $\int_V H(v)e^{\frac12 Q+P}dv$

$$\sim H(\partial)e^{P(\partial)}e^{-Q^{-1}(\varphi)/2}\Big|_{\varphi=0}$$

is



$$= \sum_{\text{Diagrams}} c(D)\begin{pmatrix}\text{Products of}\\ Q^{-1}\text{'s, } P\text{'s}\\ \text{and one } H\end{pmatrix}$$