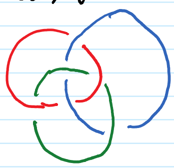
14-1100 Sep 15, hours 4-5: quotients, isomorphism theorems

September-11-14 2:45 PM

Riddle Along. Can you draw 4 linked loops, so that if you drop any one of them, the remaining 3 are not linked?



Read Along. Selick 1.1–1.4

Today's menu. Quotients and the isomorphism thms

Reminder: Given $N \triangleleft G$ ($\forall g \in G \;\; N^g = g^{-1}Ng = N$), we seek $\sim$ on G s.t. $\phi: G \to G/\sim =: H$ will be a group homomorphism with $\ker\phi = N$.

on board

$$g_1 \sim g_2 \iff \phi(g_1) = \phi(g_2) \iff \phi(g_1 g_2^{-1}) = e \iff$$
$$\iff g_1 g_2^{-1} \in N \iff g_1 \in g_2 N \iff g_1 N = g_2 N$$

Let $H = G/\sim = \{[g]\}$ where $[g] = gN$

with $\phi: G \to H$ being $\phi(g) = [g]$

define $[g_1][g_2] = [g_1 g_2]$
$[g]^{-1} = [g^{-1}]$ (well defined?)

Claim $H = G/\sim$ is a group & ϕ is a morphism whose kernel is N ... we write $H = G/N$.

Theorem (The First Isomorphism Theorem) Given any morphism $\phi: G \to H$, $G/\ker\phi \cong \operatorname{im}\phi$.

Pf construct $R: \longrightarrow$ by $[g] \longmapsto \phi(g)$
$L: \longleftarrow$ by $h \longmapsto [g]$ s.t. $\phi(g) = h$.

Aside G/H when $H < G$ & Lagrange's Thm.

Claim. For $H, K < G$, $HK < G$ iff $HK = KH$.

Pf. $\Leftarrow$ $(h_1k_1)(h_2k_2) = h_1h_2'k_2'k_2$

$\Rightarrow$ $(hk)^{-1} = h'k' = k^{-1}h^{-1}$ ....

Definition. $C_G(X) := \{g \in G : \forall x \in X \ \ g^{-1}xg = x\}$
$Z(G) := C_G(G)$
$N_G(X) := \{g \in G : g^{-1}Xg = X\}$

all are subgroups!

Claim. If $H \subset N_G(K)$ then $HK = KH$, $K \triangleleft HK$, & $H \cap K \triangleleft H$.
Pf. trivial.

The 2nd isomorphism theorem.
If $H < N_G(K)$, then

$$HK/K \cong H/H\cap K$$

$$\begin{tikzcd}
 & HK \arrow[d, dash, "\cong"] \\
H \arrow[ur, dash] \arrow[d, dash, "\cong"'] & K \\
H\cap K \arrow[ur, dash] &
\end{tikzcd}$$

Pf. R: $\rightarrow$: $[h]_K \longrightarrow [h]_{H\cap K}$ L: $\Leftarrow$: obvious.

The 3rd Isomorphism Thm.
If $K, H \triangleleft G$ & $K < H$, then $\dfrac{G/K}{H/K} \cong G/H$

Pf. R: $\rightarrow$: $[[g]_K]_{H/K} \longrightarrow [g]_H$
well defined? $[[g_1]_K]_{H/K} = [[g_2]_K]_{H/K} \Rightarrow$
$\Rightarrow [g_1]_K[g_2]_K^{-1} = [h]_K \Rightarrow g_1g_2^{-1} = hk = h$

The 4th Isomorphism Thm.
If $N \triangleleft G$ then $\pi : G \to G/N$ induces a "faithful" bijection between subgroups of G/N and $\{H : N < H < G\}$:

* $A < B \Leftrightarrow \pi(A) < \pi(B)$ (& then, $[B:A] = [\pi(B):\pi(A)]$)
* $A \triangleleft B \Leftrightarrow \pi(A) \triangleleft \pi(B)$
* $\pi(A \cap B) = \pi(A) \cap \pi(B)$.

Also did: $\operatorname{sign}(\sigma) := \operatorname{sign}\left(\prod_{i<j}(\sigma_i - \sigma_j)\right)$