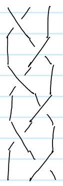
14-1100 Oct 9, hour 15: Finish semi-direct products, groups of order 12

September-27-14 1:18 PM

HW2 discussion. I should have added to HW2: $G \rtimes_{\text{conj. action}} G \cong G \times G$

Aside.



Two reasons why I like this one:
1. knotted $20's
2. Borromean.
3. It is a commutator.

Q. Can you find a 4-component Brunnian link?

Today's Menu. semi-direct products, groups of order 12 ✓

Reminders. Given N, H, $\phi: H \to \mathrm{Aut}(N)$,

$$N \rtimes H := \{nh\}; \quad n_1h_1 \cdot n_2h_2 = n_1\phi_{h_1}(n_2)h_1h_2$$

Thm 1. $N\rtimes H$ is a group, $H < N\rtimes H$, $N \triangleleft N\rtimes H$, $N \cap H = \{e\}$ $\quad (N\rtimes H/N = H)$

2. In general, if $G = NH$, $N \triangleleft G$, $H < G$, $N \cap H = \{e\}$, Then $G \cong N \rtimes_\phi H$ w/ $\phi_h(n) = hnh^{-1}$

$PB_n := \pi_1(\mathbb{C}^n \setminus \text{diags})$ = "pure braids on n strands"

$\rho: PB_n \to PB_{n-1}$ $\quad$ $\ker\rho = F_{n-1}$ and

$$PB_n = F_{n-1} \rtimes PB_{n-1} = F_{n-1} \rtimes (F_{n-2} \rtimes (\dots (F_2 \rtimes \mathbb{Z})\dots))$$

Aside.

$$B_n = \left\langle \sigma_1, \dots \sigma_{n-1} : \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j|>1 \end{array} \right\rangle$$

} an aside on free groups, generators & relations.

Groups of order 21. $\mathbb{Z}/21$, $\mathbb{Z}/7 \rtimes \mathbb{Z}/3 = \langle x \rangle \rtimes \langle y \rangle$

$\mathrm{Aut}(\mathbb{Z}/7) = \mathbb{Z}/6 = \langle \phi \rangle$; $\phi(x) = x^3$; $\quad y \mapsto \phi^0$ or ϕ^2 or ϕ^4

$$yxy^{-1} = \underbrace{x}_{\mathbb{Z}/21} \text{ or } \underbrace{x^2 \text{ or } x^4}_{\text{isomorphic}}$$

iso: if $yxy^{-1} = x^2$ & then $y^2xy^{-2} = x^4$, so

$$G_2 = \langle x \rangle \rtimes_{\phi^2} \langle y \rangle \longrightarrow \langle \bar{x} \rangle \rtimes_{\phi^4} \langle \bar{y} \rangle = G_4$$

$$\begin{pmatrix} x \\ y^2 \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{y}^{-2} \end{pmatrix} \text{ is iso.}$$

skipped

Groups of order 12. If $|G| = 12$, $P_4 = \mathbb{Z}/4$ or $(\mathbb{Z}/2)^2$, $P_3 = \mathbb{Z}/3$, and at least one of those is normal, for there's not enough room for 4 P_3 & 3 P_4's. So G is a semi-direct product: $\mathbb{Z}/4 \rtimes \mathbb{Z}/3$: must be $\mathbb{Z}/4 \times \mathbb{Z}/3 = \mathbb{Z}/12$ done ($\mathrm{Aut}(\mathbb{Z}/4) = \mathbb{Z}/2$!)

$(\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/3$: Either direct; $\mathbb{Z}/2 \times \mathbb{Z}/6$ done

or the fun action of $\mathbb{Z}_3$ on $(\mathbb{Z}/2)^2$, giving A_4 skipped

$\langle (234) \rangle$ $\quad$ e, $(12)(34)$, $(13)(24)$, $(14)(23)$

$\mathbb{Z}/3 \rtimes (\mathbb{Z}/2 \times \mathbb{Z}/2)$: Either direct or $D_6 \times \mathbb{Z}/2 = D_{12}$ done

$\mathbb{Z}/3 \rtimes \mathbb{Z}/4$: Either direct or $\mathbb{Z}/3 \rtimes \mathbb{Z}/4$ done

Solvable Groups. <u>Def</u> G is solvable if all quotients in its Jordan-Hölder series are Abelian.

<u>Thm</u> 1. If $N \triangleleft G$, G is solvable iff N & G/N are.

2. If $H < G$ and G is solvable, so is H.

$A \triangleleft B$ $\quad H \cap A \triangleleft H \cap B$? ✓ $\quad \frac{H \cap B}{H \cap A} \longrightarrow B/A$ by $[b]_{H \cap A} \to [b]_A$ is injective.

<u>Cor.</u> If a group contains A_n $n \neq 4$, it is not solvable.