14-1100 Oct 6, hours 13-14: Finish Sylow, examples, semi-direct products

September-27-14 1:18 PM

HW1 due!

Riddle Along. $\boxed{\forall \alpha \in \mathbb{R}\ \exists a_i \in \mathbb{Q} \text{ s.t. } a_i \to \alpha}$ $\boxed{\mathbb{Q} \cap [-\infty, \alpha]}$ so what?

Today's Menu. Finish Sylow, semi-direct products

Reminders. Theorem. 1. Sylow p-groups always exist; $Syl_p(G) \neq \emptyset$.

2. Every p-group is contained in a Sylow-p group.

3. All Sylow-p subgroups of G are conjugate, and

$n_p(G) := |Syl_p(G)| = 1 \bmod p$ & $n_p(G) \mid |G|$

The extension trick: Can't extend a Sylow by something of order p.

Proposition. If $P \in Syl_p(G)$, then $|\text{conjugates of } P| = 1 \bmod p$. (and $n_p \mid |G|$, of course)

Proposition. If H is a p-subgroup & $P \in Syl_p(G)$, then H is contained is a conjugate of P. [In particular, all Sylow-p subgroups are conjugates]

Proof. H acts on the set of conjugates of P by conjugation. There must be a singleton orbit — a P' s.t. $H < N_G(P')$.

Semi-Direct Products. If $N<G$, $H<G$, compare $N \times H$ with NH.

There's always $\mu: N \times H \to NH$ by $(n,h) \mapsto nh$.

In general, nothing to say.

If $N \cap H = \{e\}$, injective but image might not be a group.

Example: $\langle(123)\rangle, \langle(345)\rangle \subset S_5$

If $N \cap H = \{e\}$ & $N \lhd G$ & $H \lhd G$, then $[N,H]=\{e\}$ &

$NH \cong N \times H$.

The interesting case is when $N \cap H = \{e\}$, $N \lhd G$, H maybe not.

Get $H \xrightarrow{\phi} Aut(N)$ by $h \mapsto (n \mapsto n^{h^{-1}} = h n h^{-1})$

or $\phi_h(n) = h n h^{-1}$

$n_1h_1n_2h_2 = n_1h_1n_2h_1^{-1}h_1h_2 = n_1\phi_{h_1}(n_2)\,h_1h_2$

$(nh)^{-1} = h^{-1}n^{-1} = h^{-1}n^{-1}hh^{-1} = \phi_{h^{-1}}(n^{-1})\cdot h^{-1}$

Definition. Given abstract N, H & $\phi: H \to \mathrm{Aut}(N)$, the semi-direct product $N \rtimes H$.

Prop. 1. In the above case, $\mu: N\rtimes H \to NH$ is an isomorphism.

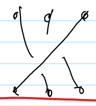
2. $H < N\rtimes H$, $N \triangleleft (N\rtimes H)$ and $N\rtimes H/N \cong H$.

Small Examples. 1. $D_{2n} = \mathbb{Z}/n \rtimes \{\pm 1\}$

2. $\{ax+b\} = \mathbb{R}^+_b \rtimes \mathbb{R}^\times_a$

3. $\{Ax+b : A\in GL(V), b\in V\} = V_b \rtimes GL(V)_A$

4. "The Poincare/Relativity Group" $= \mathbb{R}^4 \rtimes SO(3,1)$

Big Example. $B_n = \pi_1((\mathbb{C}^2 - \{\text{diags}\})/S_n) =$ 

I should have started the discussion of PB_n w/ an intro to free groups and w/ $\pi_1(\text{xxx}) = F_n$

done line

$B_n = \left\langle \sigma_1, \dots \sigma_{n-1} : \begin{matrix} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j|>1 \end{matrix} \right\rangle$ } an aside on free groups, generators & relations.

$\pi: B_n \longrightarrow S_n \qquad PB_n = \ker \pi$

$PB_n \triangleleft B_n$ yet not $B_n = PB_n \rtimes S_n$

Two reasons why I like this one: 1. knotted $20's 2. Borromean.

$\rho: PB_n \longrightarrow PB_{n-1} \qquad \ker\rho = F_{n-1}$ and

$PB_n = F_{n-1} \rtimes PB_{n-1} = F_{n-1} \rtimes (F_{n-2} \rtimes (\dots (F_2 \rtimes \mathbb{Z})..))$

Groups of order 21. $\mathbb{Z}/21$, $\mathbb{Z}/7 \rtimes \mathbb{Z}/3 = \langle x \rangle \rtimes \langle y \rangle$

$\mathrm{Aut}(\mathbb{Z}/7) = \mathbb{Z}/6 = \langle \phi_3 \rangle$; $\phi_3(x) = x^3$; $x^y = x$ or x^2 or x^4 (isomorphic)

(iso: if $x^y = x^2$ & $\bar{y} = y^2$ then $x^{\bar{y}} = x^4$)

Groups of order 12. If $|G| = 12$, $P_4 = \mathbb{Z}/4$ or $(\mathbb{Z}/2)^2$, $P_3 = \mathbb{Z}/3$, and at least one of those is normal, for there's not enough room for 4 P_3 & 3 P_4's. So G is a semi-direct product: $\mathbb{Z}/4 \rtimes \mathbb{Z}/3$: must be $\mathbb{Z}/4 \times \mathbb{Z}/3 = \mathbb{Z}/12$ ($\mathrm{Aut}(\mathbb{Z}/4) = \mathbb{Z}/2$!)

$(\mathbb{Z}/2\times\mathbb{Z}/2)\rtimes\mathbb{Z}/3$: Either direct; $\mathbb{Z}/2 \times \mathbb{Z}/6$

or the fun action of $\mathbb{Z}_3$ on $(\mathbb{Z}/2)^2$, giving A_4

$\langle (234) \rangle$

e
$(12)(34)$
$(13)(24)$
$(14)(23)$

$\mathbb{Z}/3 \rtimes (\mathbb{Z}/2 \times \mathbb{Z}/2)$: Either direct or $D_6 \times \mathbb{Z}/2 = D_{12}$

$\mathbb{Z}/3 \rtimes \mathbb{Z}/4$: Either direct or $\mathbb{Z}/3 \rtimes \mathbb{Z}/4$