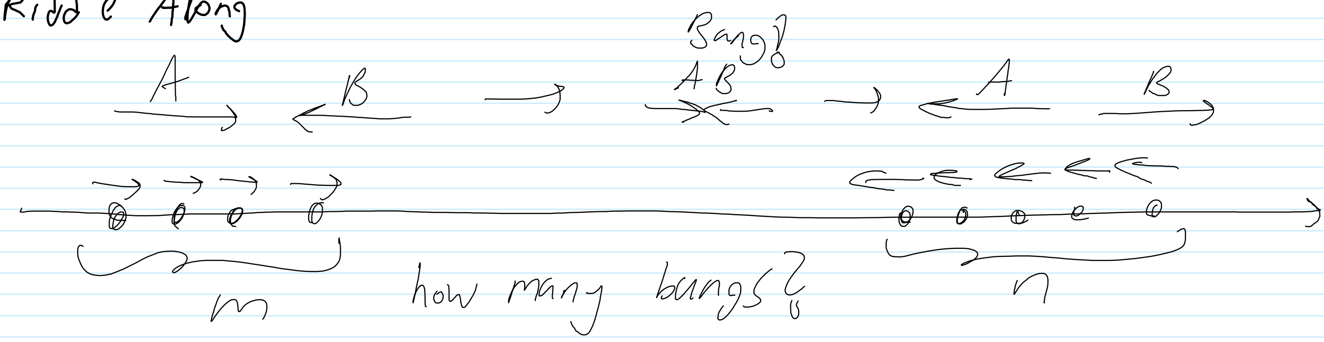
14-1100 Oct 23, hour 19: Solvable groups, rings

October-10-14 1:09 PM

Return TT, etc.

HW3 on Web, more may be added next week.

Riddle Along



Solvable Groups. Def G is solvable if all quotients in its Jordan-Hölder series are Abelian.

Thm 1. If $N \triangleleft G$, G is solvable iff N & G/N are.

2. If $H < G$ and G is solvable, so is H.

$A \triangleleft B$ $\quad H \cap A \triangleleft H \cap B$? ✓ $\quad \frac{H \cap B}{H \cap A} \longrightarrow B/A$ by $[b]_{H \cap A} \to [b]_A$ is injective.

Cor. If a group contains A_n $n \geq 5$, it is not solvable.

Rings.

Definition 2.1.1. *A* ***ring*** *consists of a set R together with binary operations $+$ and $\cdot$ satisfying:*

1. *$(R, +)$ forms an abelian group,*
2. *$(a \cdot b) \cdot c = a \cdot (b \cdot c)$ $\forall a, b, c \in R$,*
3. *$\exists 1 \neq 0 \in R$ such that $a \cdot 1 = 1 \cdot a = a$ $\forall a \in R$, and*
4. *$a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ $\forall a, b, c \in R$.*

Also define: Commutative ring.

Examples. $\mathbb{Z}$, $R[x]$, $M_{n \times n}(R)$, RG

Morphisms, Examples: 1. $\mathbb{Z} \longrightarrow \mathbb{Z}/n$

2. $R \longrightarrow R[x]$ at deg 0

3. $R \to M_{n \times n}(R)$ as diag

4. $ev_u : R[x] \to R$ (if R is commutative)

done line

5. $M_{n \times n}(R[x]) \cong M_{n \times n}(R)[x]$

6. If $\varphi : G \to H$, $\varphi_* : RG \to RH$

Cayley-Hamilton A matrix annihilates its characteristic poly:

Let $A \in M_{n \times n}(R)$, R commutative. Set

$\chi_A(t) = \det(tI - A)$. Then $\chi_A(A) = 0$

Wrong proof. $\chi_A(A) = \det(AI - A) = \det(0) = 0$

Nonesense! Would have worked for trace just as well! $\chi_A^{tr} = tr(tI - A) = nt - tr(A)$

so $A = \frac{trA}{n} I$

The issue:

$$\begin{tikzcd} M_{n\times n}(R)[t] \arrow[r, "\det"] \arrow[d, "ev_A"] & R[t] \arrow[d, "ev_A"] \\ M_{n\times n}(R) \arrow[r, "?"] & M_{n\times n}(R) \end{tikzcd}$$

Right proof.

in $M_{n\times n}(R[t])$ in $M_{n\times n}(R)[t]$

$\det(tI - A)\cdot I = adj(tI - A)(tI - A) = (\sum B_i t^i)(tI - A)$ in

now substitute $t = A$. The B_i's commute with A

because $(tI - A)\,adj(tI - A) = adj(tI - A)(tI - A)$.

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im, subring, ker, ideal.

Q. Is every ideal a quotient.

Ans. Define R/I.