14-1100 Oct 2, hour 12: The Sylow theorems, II

September-27-14 1:18 PM

Riddle Along. Your turn again!

Today's Menu. G_{pq}, proofs of Sylow 2–3.

Reminders. Theorem. 1. Sylow p-groups always exist; $Syl_p(G) \neq \emptyset$.

2. Every p-group is contained in a Sylow-p group.

3. All Sylow-p subgroups of G are conjugate, and

$n_p(G) := |Syl_p(G)| = 1 \mod p$ & $n_p(G) \mid |G|$

Groups of order 15. $P_3 \lhd G$, $P_5 \lhd G$, $G = P_3 \times P_5 = \mathbb{Z}/3 \times \mathbb{Z}/5 = \mathbb{Z}/15$

Groups of order 21. $P_7 \lhd G$, P_3 may not be normal

If normal, $G = P_3 \times P_7 = \mathbb{Z}/21$.

Otherwise, $P_7 = \langle x \rangle$, $P_3 = \langle y \rangle$,

we have $x^y = x$, or x^2, or x^4!

Debt. What does this mean?

Aside. $\mathrm{Aut}(\mathbb{Z}/p) \overset{\shortparallel}{} (\mathbb{Z}/p)^*$ is cyclic;

$\mathrm{Aut}(\mathbb{Z}/7) = \langle x \mapsto x^3 \rangle$

skip

Groups of order pq. $n_p \mid pq \Rightarrow n_p \mid q$, (or $n_p = 1$)

$n_p = 1 \mod p \Rightarrow q = 1 \mod p \Rightarrow p \mid q-1$

If $p < q$, $p \nmid q-1 \Rightarrow G = \mathbb{Z}/pq$

if $p \mid q-1$, small may act on big....

The "extension lemma":

Lemma. 1. If $P \in Syl_p(G)$ & $H < N_G(P)$ is a p-group,

Then $H \subset P$

2. If $P \in Syl_p(G)$, $|x| = p^\beta$, $x \in N_G(P)$, Then $x \in P$.

Reformulation: $P \in Syl_p(G)$, $|H| = p^\beta \Rightarrow N_H(P) = H \cap P$

Proposition. If $P \in Syl_p(G)$, then $|\text{conjugates of } P| = 1 \mod p$.

(and $n_p \mid |G|$, of course)

Proof. P acts on the set of its conjugates by conjugation. The orbit

$\{P\}$ is a singleton; by lemma, the sizes of all other orbits are divisible by p. done line

Proposition. If H is a p-subgroup & $P \in Syl_p(G)$, then H is contained is a conjugate of P. [In particular, all Sylow-p subgroups are conjugates]

Proof. H acts on the set of conjugates of P by conjugation. There must be a singleton orbit — a P' s.t. $H < N_G(P')$.