Do not turn this page until instructed.

Math 1100 Core Algebra I

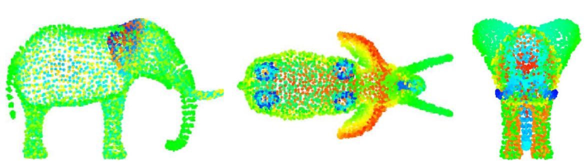
Term Test

University of Toronto, October 20, 2014

Solve 3 of the 4 problems on the other side of this page.
Each problem is worth 33 points, and you get one point for free.
You have an hour and fifty minutes to write this test.

Notes

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and made of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.
- Advance apology: It may take me some time to grade this exam; sorry.
- Barring the unforeseen, our final exam will take place on Monday December 15, 12-3PM, at Bahen 6183.



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Good Luck!

Solve 3 of the following 4 problems. Each problem is worth 33 points. You have an hour and fifty minutes. **Neatness counts! Language counts!**

Problem 1. Let n be a natural number and let F be a subset of the set $\{(i, j)\colon 1 \le i \le j \le n\}$ that contains the diagonal $\{(i, i)\}$. For each $(i, j) \in F$ you are given an element $\sigma_{i,j}$ of the permutation group S_n having the property that $\sigma_{i,j}(\alpha) = \alpha$ if $\alpha < i$, and $\sigma_{i,j}(i) = j$. Assume also that for every i, $\sigma_{i,i} = \iota$, the identity permutation. Let M_1 be the set of all "monotone products":

$$M_1 := \left\{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n} \colon \forall i\ (i, j_i) \in F\right\}.$$

1. It is also given that for every $(i, j) \in F$ and every $(k, l) \in F$, we have $\sigma_{i,j}\sigma_{k,l} \in M_1$. Prove that M_1 is a subgroup of S_n.
2. In one or two paragraphs, explain why we cared about this statement in class. What did it give us that we could not have had without it?

Problem 2. Let G be a group and let $Z(G)$ denote its centre.

1. Show that if $G/Z(G)$ is cyclic then G is Abelian.
2. Prove that if the group $\mathrm{Aut}(G)$ of automorphisms of G is cyclic, then G is Abelian.

Problem 3. Let G be a finite group, let p be a prime number, let α be the largest natural number such that $p^\alpha \mid |G|$, and let P be a subgroup of G whose order is p^α.

1. Suppose that $x \in G$ is an element whose order is a power of p, and suppose that x normalizes P. Show that $x \in P$.
2. Prove that the number of conjugates of P in G is 1 modulo p. (You are not allowed to use the Sylow theorems, of course).

Problem 4. Let H and N be group, and let $\phi\colon H \to \mathrm{Aut}(N)$ be given. Remember that we may consider H and N as subgroups of the semi-direct product $N \rtimes H := N \rtimes_\phi H$.

1. Prove that $(N \rtimes H)/N = H$.
2. Prove that the centralizer of N within H (that is, the set of elements of H that commute with every element of N) is $\ker\phi$.
3. Prove that the centralizer of H within N is equal to the normalizer of H within N (where the latter is the set of $n \in N$ such that $nHn^{-1} = H$).

Good Luck!