Do not turn this page until instructed.

Math 1100 Core Algebra I

Final Examination

University of Toronto, December 15, 2014

Solve the 5 of the 6 problems on the other side of this page.
Each problem is worth 20 points.
You have three hours to write this test.

Notes.

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and made of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.

Good Luck!

Solve 5 of the following 6 problems. Each problem is worth 20 points. You have three hours. **Neatness counts! Language counts!**

Problem 1. Prove that a ring R is a PID iff it is a UFD in which $\gcd(a, b) \in \langle a, b\rangle$ for every non-zero $a, b \in R$.

Correction. You may use the result "PID⇒UFD" without proof.

Problem 2. Let G be a finite group, let p be a prime number, and let P be a Sylow-p subgroup of G.

1. Suppose that $x \in G$ is an element whose order is a power of p, and suppose that x normalizes P. Show that $x \in P$.
2. Prove that the number of conjugates of P in G is 1 modulo p. (You are not allowed to use the Sylow theorems, of course).

Problem 3. Let M and N be some modules over a ring R.

1. Define $M \otimes N$ as the solution of some universal property.
2. Prove that if such a solution exists, it is unique up to an isomorphism.
3. Using only this universal-property definition, prove that $M \otimes R \cong M$.

Problem 4. Prove the following simplified version of the structure theorem for finitely generated modules over a PID:

Let R be a PID and let M be the R-module $R^n/\langle r_1, \ldots, r_m\rangle$, where n and m are natural numbers and $r_1, \ldots, r_m \in R^n$. Then there exists a natural number k and elements $a_1, \ldots, a_l$ of R so that $M \cong R^k \oplus \bigoplus_{i=1}^{l} R/\langle a_i\rangle$.

Problem 5. Let $V = F^n$ be an n-dimensional vector space over some field F, let $T\colon V \to V$ be a linear transformation, let $R = F[x]$ denote the ring of polynomials in a variable x with coefficients in F, and consider V as an R-module by setting $xv = Tv$ for any $v \in V$. Let $\pi\colon R^n \to F^n$ be the morphism of R-modules defined by mapping the standard basis elements e_i of R^n to their obvious counterparts in F^n. Propose a set of n generators r_i of $\ker \pi$ and prove in detail that your proposed r_i indeed generate $\ker \pi$.

Problem 6. Let H and K be subgroups of some group G. Prove that the left G-sets G/H and G/K are isomorphic (as left G-sets) iff the subgroups H and K are conjugate.

Hint. The map $G/H \to G/K$ is induced by multiplication on the right by some element $y \in G$. What could y be, if H and K are conjugate? How would you recover such a y, given an isomorphism of left G-sets $\phi\colon G/H \to G/K$?

Good Luck!