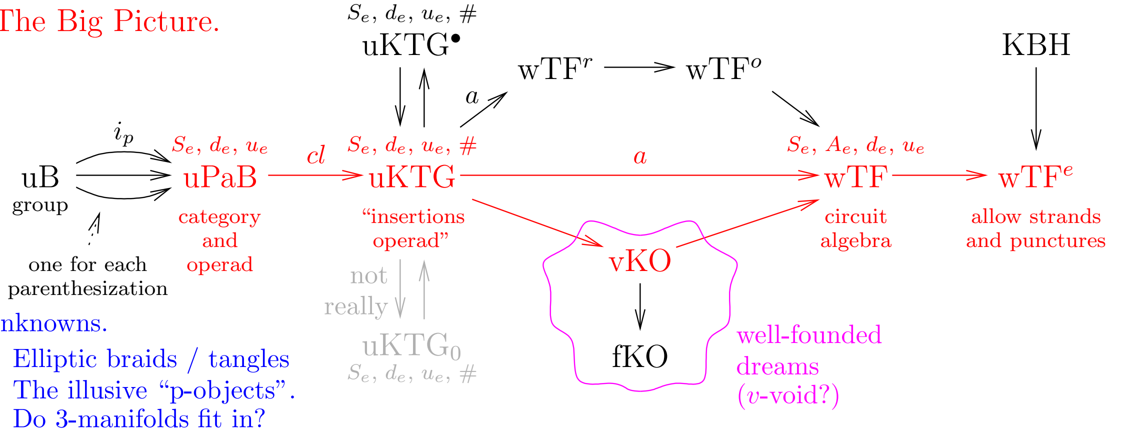
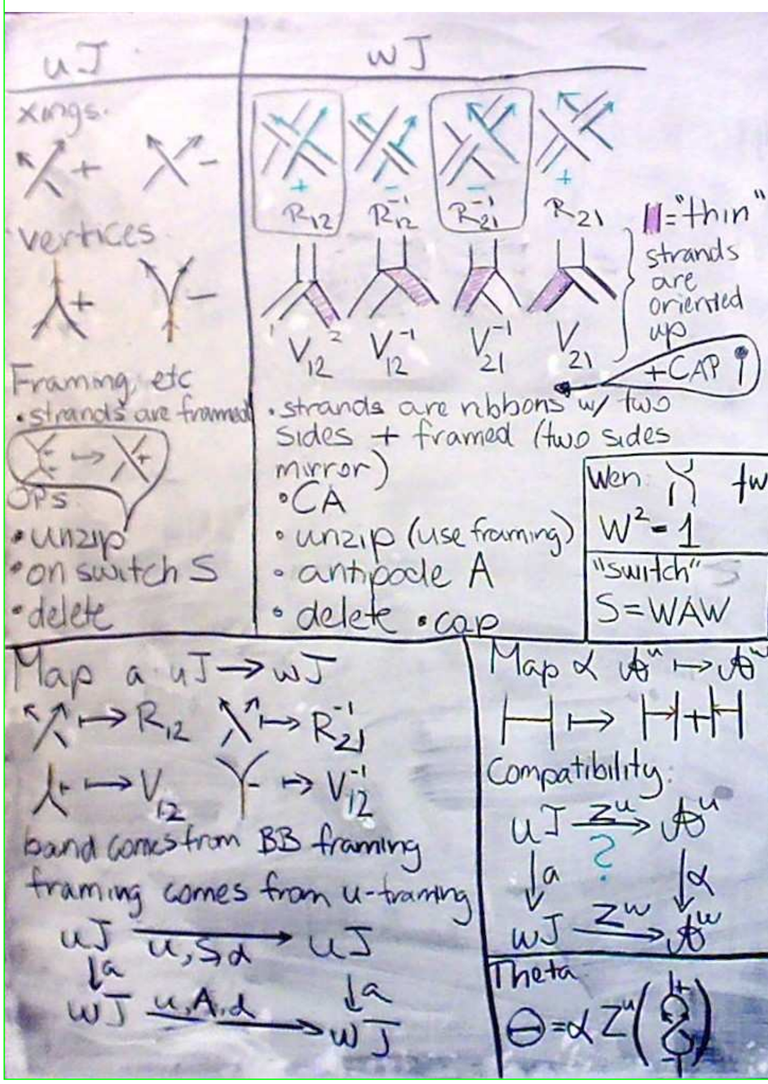
The u-Universe and the w-World

The Big Picture.

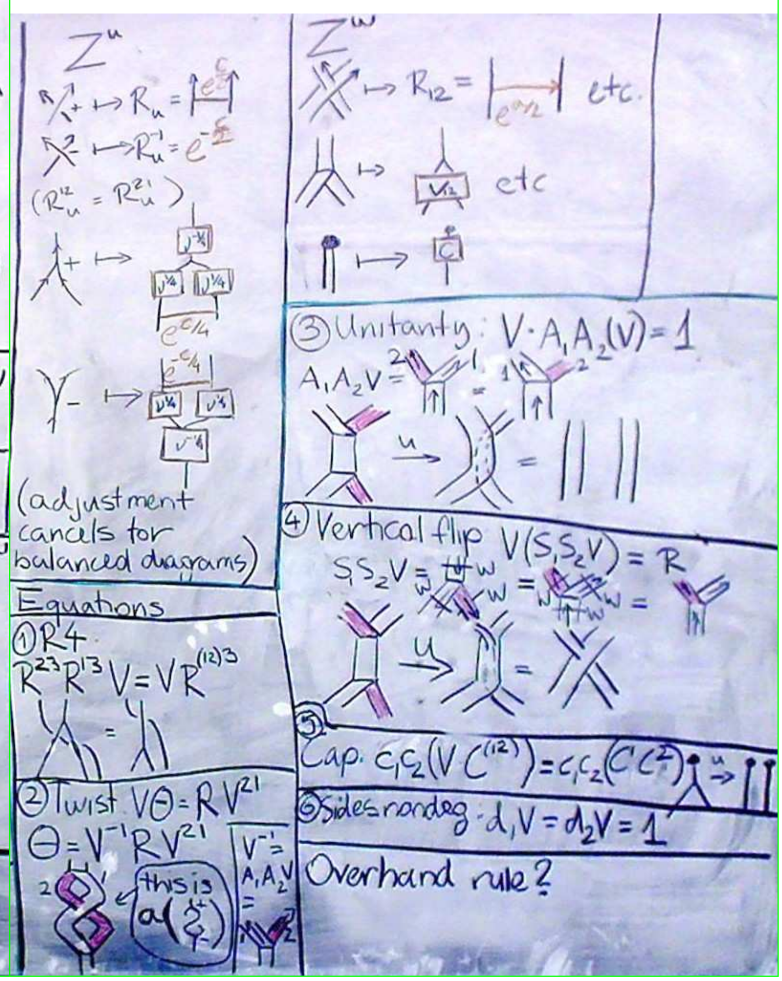


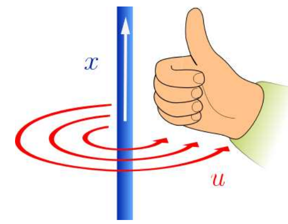
Unknowns.
1. Elliptic braids / tangles
2. The illusive "p-objects".
3. Do 3-manifolds fit in?

The u-Universe. uKTG = PA ⟨uGens |uRels| uOps⟩$_{1,1}$



The w-World. wTF = CA ⟨wGens |wRels| uOps⟩

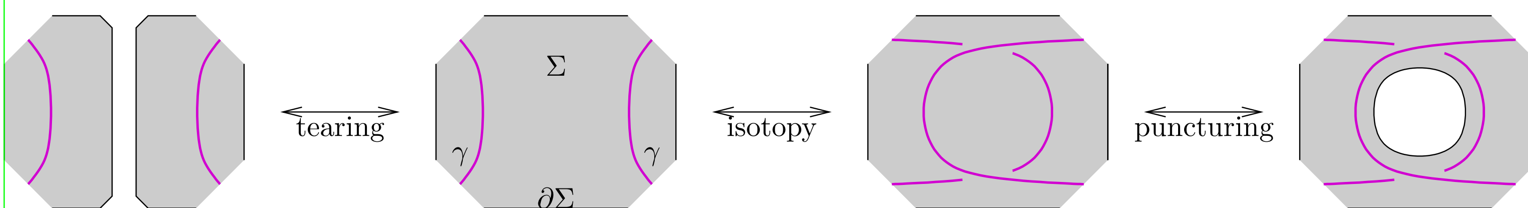




If x is an oriented S^1 and u is an oriented S^2 in an oriented S^4 (or $\mathbb{R}^4$) and the two are disjoint, their linking number l_{ux} is defined as follows. Pick a ball B whose oriented boundary is u (using the "outward pointing normal" convention for orienting boundaries), and which intersects x in finitely many transversal intersection points p_i. At any of these intersection points p_i, the concatenation of the orientation of B at p_i (thought of a basis to the tangent space of B at p_i) with the tangent to x at p_i is a basis of the tangent space of S^4 at p_i, and as such it may either be positively oriented or negatively oriented. Define $\sigma(p_i) = +1$ in the former case and $\sigma(p_i) = -1$ in the latter case. Finally, let $l_{ux} := \sum_i \sigma(p_i)$. It is a standard fact that l_{ux} is an isotopy invariant of (u, x).
An efficient thumb rule for deciding the linking-number signs for a balloon u and a hoop x presented using our standard notation is the "right-hand rule" of the figure on the right, shown here without further explanation. The lovely figure is adopted from [Wikipedia: Right-hand rule].

v-Knots are oriented knots drawn on an oriented surface Σ (meaning, "embedded in $\Sigma \times [-\epsilon, \epsilon]$"), modulo "stabilization", which is the addition and/or removal of empty handles (handles that do not intersect with the knot). We prefer an equivalent, yet even more bare-bones approach. For us, a virtual knot is an oriented knot γ drawn on a "virtual surface Σ for γ". More precisely, Σ is an oriented surface that may have a boundary, γ is drawn on Σ, and the pair (Σ, γ) is taken modulo the following relations:

- Isotopies of γ on Σ (meaning, in $\Sigma \times [-\epsilon, \epsilon]$).
- Tearing and puncturing parts of Σ away from γ:



(We call Σ a "virtual surface" because tearing and puncturing imply that we only care about it in the immediate vicinity of γ).
We can now define a map δ_0, defined on v-knots and taking values in ribbon tori in $\mathbb{R}^4$: given (Σ, γ), embedd Σ arbitrarily in $\mathbb{R}^3_{xyz} \subset \mathbb{R}^4$. Note that the unit normal bundle of Σ in $\mathbb{R}^4$ is a trivial circle bundle and it has a distiguished trivialization, constructed using its positive-t-direction section and the orientation that gives each fiber a linking number $+1$ with the base Σ. We say that a normal vector to Σ in $\mathbb{R}^4$ is "near unit" if its norm is between $1-\epsilon$ and $1+\epsilon$. The near-unit normal bundle of Σ has as fiber an annulus that can be identified with $[-\epsilon, \epsilon] \times S^1$ (identifying the radial direction $[1-\epsilon, 1+\epsilon]$ with $[-\epsilon, \epsilon]$ in an orientation-preserving manner), and hence the near-unit normal bundle of Σ defines an embedding of $\Sigma \times [-\epsilon, \epsilon] \times S^1$ into $\mathbb{R}^4$. On the other hand, γ is embedded in $\Sigma \times [-\epsilon, \epsilon]$ so $\gamma \times S^1$ is embedded in $\Sigma \times [-\epsilon, \epsilon] \times S^1$, and we can let $\delta_0(\Sigma, \gamma)$ be the composition
$$\gamma \times S^1 \hookrightarrow \Sigma \times [-\epsilon, \epsilon] \times S^1 \hookrightarrow \mathbb{R}^4,$$
which is a torus in $\mathbb{R}^4$, oriented using the given orientation of γ and the standard orientation of S^1.