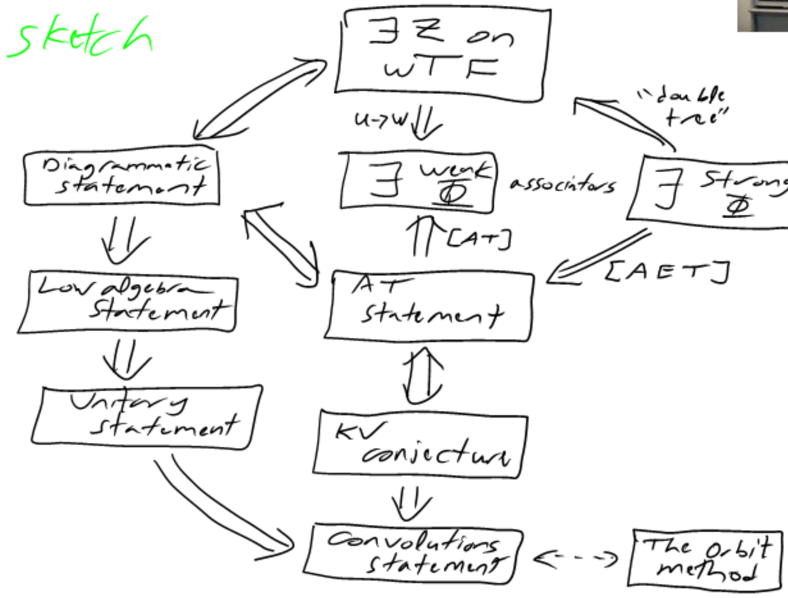
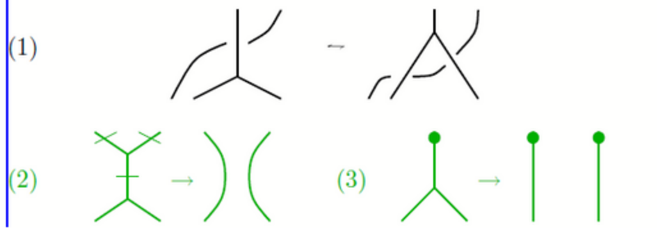
Clippings from http://www.math.toronto.edu/~drorbn/Talks/{Bonn-0908, Goettingen-1004, Montpellier-1006}

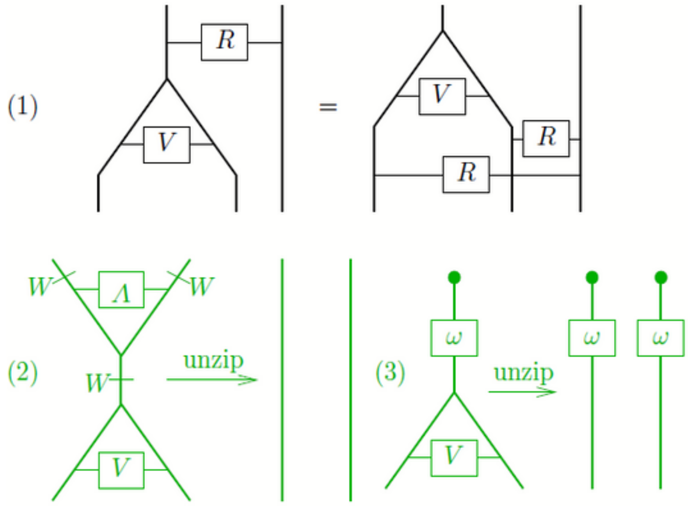
Yesterday's stuff (KBH) is in the nearly-finished http://www.math.toronto.edu/~drorbn/papers/KBH/; today's is in the nearly finished http://drorbn.net/?title=WKO (joint with Z. Dancso)



Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp \mathsf{H} \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



Alekseev-Torossian statement. There is an element $F \in \mathrm{TAut}_2$ with

$$F(x+y) = \log e^x e^y$$

and $j(F) \in \operatorname{im}\tilde\delta \subset \mathrm{tr}_2$, where for $a \in \mathrm{tr}_1$, $\tilde\delta(a) := a(x) + a(y) - a(\log e^x e^y)$.

Alekseev Torossian

Free Lie statement (Kashiwara-Vergne). There exist convergent Lie series F and G so that with $z = \log e^x e^y$

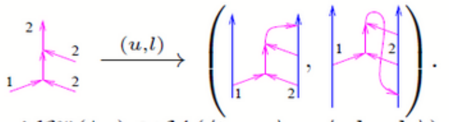
$$x + y - \log e^y e^x = (1 - e^{-\operatorname{ad} x})F + (e^{\operatorname{ad} y} - 1)G$$

$$\mathrm{tr}(\operatorname{ad} x)\partial_x F + \mathrm{tr}(\operatorname{ad} y)\partial_y G = \frac{1}{2}\mathrm{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1} + \frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1} - \frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1} - 1\right)$$

Wheels and Trees. With $\mathcal{P}$ for Primitives,

$$0 \longrightarrow \langle\text{wheels}\rangle \overset{\iota}{\longrightarrow} \mathcal{P}\mathcal{A}^w(\uparrow_n) \underset{l}{\overset{u}{\underset{\pi}{\rightleftarrows}}} \langle\text{trees}\rangle \longrightarrow 0,$$

with



So $\operatorname{proj}\mathcal{K}^w(\uparrow_n) \cong \mathcal{U}(\langle\text{trees}\rangle \ltimes \langle\text{wheels}\rangle)$.

trees atop a wheel, and a little prince

Some A-T Notions. $\mathfrak{a}_n$ is the vector space with basis $x_1, \dots, x_n$, $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ is the free Lie algebra, $\mathrm{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ is the free associative algera "of words", $\mathrm{tr} : \mathrm{Ass}_n^+ \to \mathrm{tr}_n = \mathrm{Ass}_n^+/(x_{i_1}x_{i_2}\cdots x_{i_m} = x_{i_2}\cdots x_{i_m}x_{i_1})$ is the "trace" into "cyclic words", $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ are all the derivations, and

$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n : \forall i\ \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

are "tangential derivations", so $D \leftrightarrow (a_1, \dots, a_n)$ is a vector space isomorphism $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_n \mathfrak{lie}_n$. Finally, $\operatorname{div} : \mathfrak{tder}_n \to \mathrm{tr}_n$ is $(a_1, \dots, a_n) \mapsto \sum_k \mathrm{tr}(x_k(\partial_k a_k))$, where for $a \in \mathrm{Ass}_n^+$, $\partial_k a \in \mathrm{Ass}_n$ is determined by $a = \sum_k (\partial_k a)x_k$, and $j : \mathrm{TAut}_n = \exp(\mathfrak{tder}_n) \to \mathrm{tr}_n$ is $j(e^D) = \frac{e^D - 1}{D} \cdot \operatorname{div} D$.

Theorem. Everything matches. $\langle\text{trees}\rangle$ is $\mathfrak{a}_n \oplus \mathfrak{tder}_n$ as Lie algebras, $\langle\text{wheels}\rangle$ is tr_n as $\langle\text{trees}\rangle$ / $\mathfrak{tder}_n$-modules, $\operatorname{div} D = \iota^{-1}(u - l)(D)$, and $e^{uD}e^{-lD} = e^{jD}$.

Differential Operators. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\operatorname{ad} x$: $(x\varphi)(y) := \varphi([x, y])$.

Trees become vector fields and $uD \mapsto lD$ is $D \mapsto D^*$. So $\operatorname{div} D$ is $D - D^*$ and $jD = \log(e^D(e^D)^*) = \int_0^1 dt e^{tD} \operatorname{div} D$.

Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{\mathcal{U}}(I\mathfrak{g})$, with W the automorphism of $\hat{\mathcal{U}}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $\omega \in \hat{\mathcal{S}}(\mathfrak{g}^*)$ and $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that

(1) $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$

(2) $V \cdot SWV = 1$ (3) $(c \otimes c)(V\Delta(\omega)) = \omega \otimes \omega$

Unitary statement. There exists $\omega \in \mathrm{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that

(1) $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

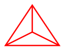
(2) $VV^* = I$ (3) $V\omega_{x+y} = \omega_x\omega_y$

Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j : \mathfrak{g} \to \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \to G$, and let $\Phi : \mathrm{Fun}(G) \to \mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$. Then if $f, g \in \mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then

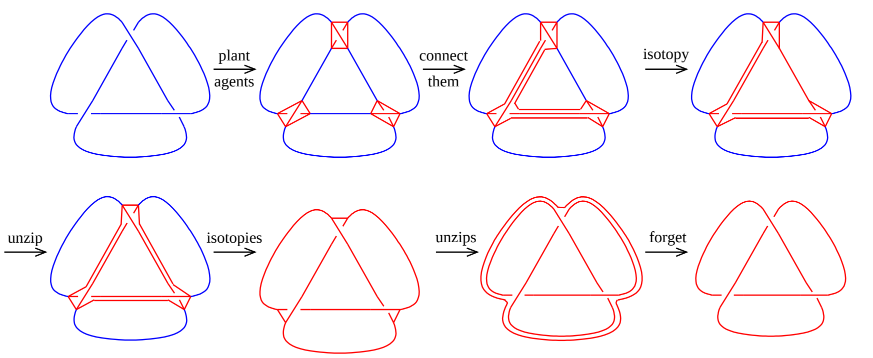
$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

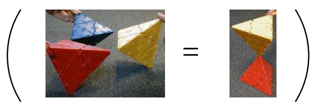
The Orbit Method. By Fourier analysis, the characters of $(\mathrm{Fun}(\mathfrak{g})^G, \star)$ correspond to coadjoint orbits in $\mathfrak{g}^*$. By averaging representation matrices and using Schur's lemma to replace intertwiners by scalars, to every irreducible representation of G we can assign a character of $(\mathrm{Fun}(G)^G, \star)$.

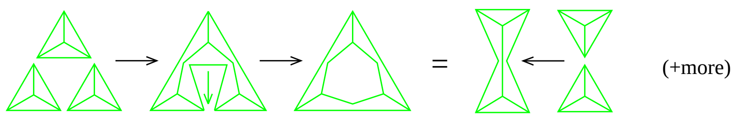
[From DBN/Talks/Montpellier-1006]

Using moves, KTG is generated by ribbon twists and the tetrahedron  :

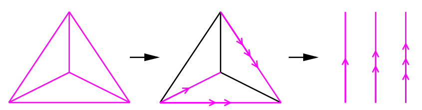
All strands here are green

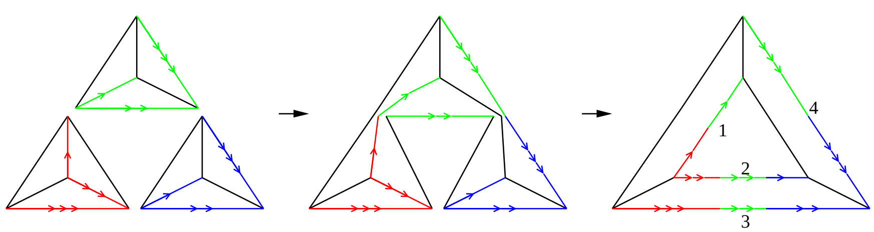


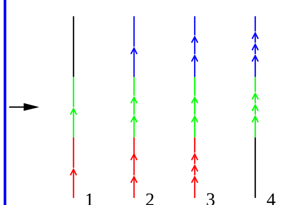
Modulo the relation(s): 



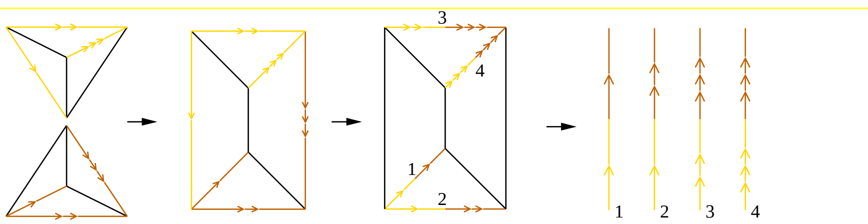
Claim. With $\Phi := Z(\text{tetrahedron})$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.

Proof.  $:= \Phi \in \mathcal{A}(\uparrow_3)$



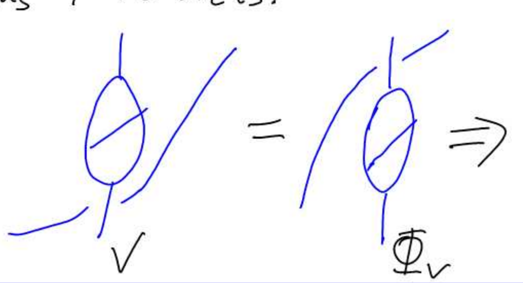


$= (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (1 \otimes \Phi) \in \mathcal{A}(\uparrow_4)$



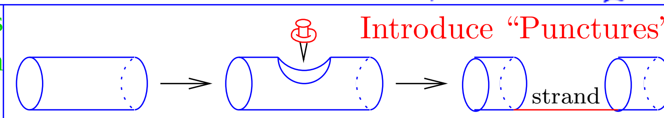
$= (\Delta \otimes 1 \otimes 1)(\Phi) \cdot (1 \otimes 1 \otimes \Delta)(\Phi) \in \mathcal{A}(\uparrow_4)$

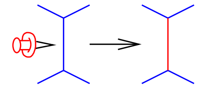
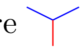
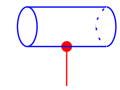
{SolKV} → {Associators}: Trivial – a tetrahedron has 4 vertices.

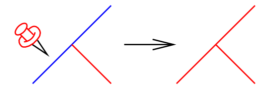


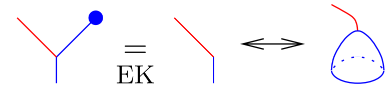
$\Phi_V \in \mathfrak{sder}_3$, so it is a sum of unrooted trees.

Introduce "Punctures".



So  where  means .

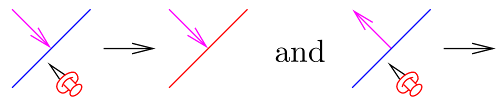
Punctures expand to the nearest Y-vertex: 

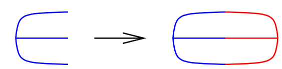
Note. 

$\mathcal{K}^{\overline{w}}$. Allow tubes and strands and tube-strand vertices as above, yet allow only "compact" knots — nothing runs to ∞.

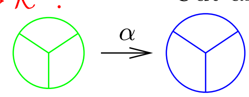
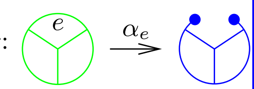
$\mathcal{K}^w \leftrightarrow \mathcal{K}^{\overline{w}}$ equivalence. $\mathcal{K}^w$ has a homomorphic expansion iff $\mathcal{K}^{\overline{w}}$ has a homomorphic expansion.

$\Longrightarrow$ Puncture $\mathcal{A}$ and Z:



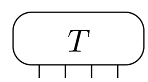
$\Longleftarrow$  (makes sense, and CA ops can be emulated)

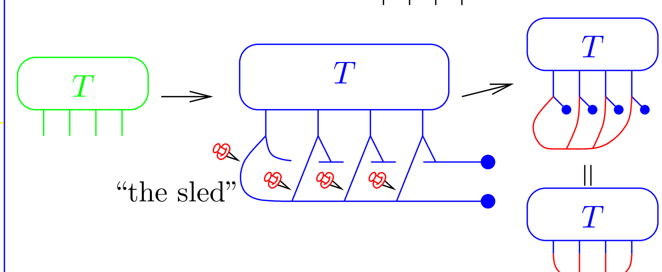
$\mathcal{K}^u \to \mathcal{K}^{\overline{w}}$. "Cut and cap is well-defined on u"

Light:  Better: 

Theorem. The generators of $\mathcal{K}^{\overline{w}}$ can be written in terms of the generators of $\mathcal{K}^u$ (i.e., given Φ, can write a formula for V.

Sketch. 

so enough to write any . Here go:



Theorem. The Alekseev-Torossian statement is equivalent to the knot-theoretic statement.

Proof. Write $V = e^c e^{uD}$ with $c \in \mathfrak{tr}_2$, $D \in \mathfrak{tder}_2$, and $\omega = e^b$ with $b \in \mathfrak{tr}_1$. Then (1) $\Leftrightarrow e^{uD}(x+y)e^{-uD} = \log e^x e^y$,
(2) $\Leftrightarrow I = e^c e^{uD}(e^{uD})^* e^c = e^{2c} e^{jD}$, and
(3) $\Leftrightarrow e^c e^{uD} e^{b(x+y)} = e^{b(x)+b(y)} \Leftrightarrow e^c e^{b(\log e^x e^y)} = e^{b(x)+b(y)}$
$\Leftrightarrow c = b(x) + b(y) - b(\log e^x e^y)$.