Three Basic Problems

October-08-08
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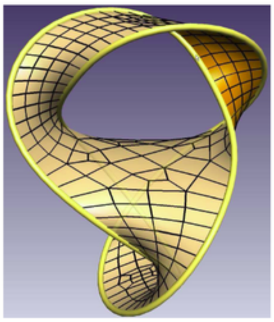
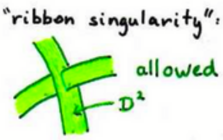
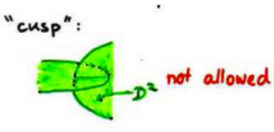
1. Determine the "genus" of a knot.
2. Determine the "unknotting number" of a knot.
3. Decide if a knot is "Ribbon":

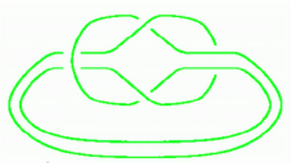


Drawn using SeifertView, http://www.win.tue.nl/~vanwijk/seifertview/





(image by Zsuzsanna Dancso)



Ribbon

Not

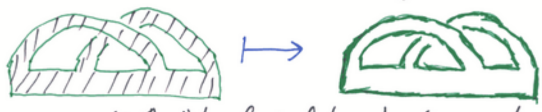
Claim 1

$$\mathcal{K}(\bigcirc)\supset\left\{\begin{matrix}\text{knots bounding}\\ \text{a surface of}\\ \text{genus 1}\end{matrix}\right\}=\{\partial_T\gamma:\gamma\in\mathcal{K}(\frown\!\frown)\}$$

knottings of a band-graph

where

∂_T:



is the "topological boundary" operator.

Algebraic Knot Theory:

Suppose we had invariants Z:

$$\begin{tikzcd}\mathcal{K}(\frown\!\frown) \arrow[r, mapsto, "Z"] \arrow[d, "\partial_T"] & \mathcal{A}(\frown\!\frown) \arrow[d, "\partial_A"] \\ \left\{\begin{matrix}\text{genus 1}\\ \text{knots}\end{matrix}\right\}=\operatorname{im}\partial_T<\mathcal{K}(\bigcirc) \arrow[r, mapsto, "Z"] & \mathcal{A}(\bigcirc)\end{tikzcd}$$

Then $Z\left(\begin{matrix}\text{genus 1}\\ \text{knots}\end{matrix}\right)\subset\operatorname{im}\partial_A$ and we have an algebraic invariant detecting $\{\text{genus}\geq1\}$.

Similarly for detecting $\{\text{genus}\geq 7\}\ldots$

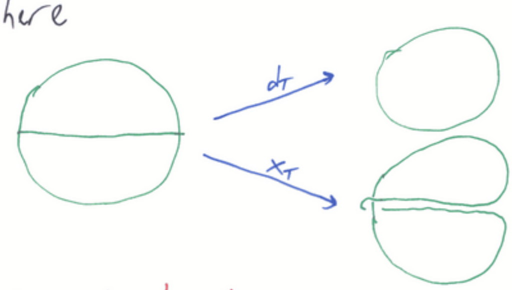
Claim 2

$$\mathcal{K}(\bigcirc)\supset\left\{\begin{matrix}\text{knots of unknotting}\\ \text{number 1}\end{matrix}\right\}=\left\{x\gamma:\begin{matrix}\gamma\in\mathcal{K}(\ominus);\\ d\gamma=\bigcirc\end{matrix}\right\}$$

knottings of $\ominus$

the unknot.

where



Algebraic Knot Theory:

$$\begin{tikzcd} & \mathcal{K}(\bigcirc) \arrow[rr, "Z"] & & \mathcal{A}(\bigcirc) \\ \mathcal{K}(\ominus) \arrow[ur, "x_T"] \arrow[rr, "Z"] \arrow[dr, "d_T"'] & & \mathcal{A}(\ominus) \arrow[ur, "x_A"] \arrow[dr, "d_A"] & \\ & \mathcal{K}(\bigcirc) \arrow[rr, "Z"] & & \mathcal{A}(\bigcirc)\ni\bigcirc\end{tikzcd}$$

So

$$Z\left(\left\{\begin{matrix}\text{unknotting}\\ \text{number 1}\end{matrix}\right\}\right)\subset\left\{x_A\alpha:\begin{matrix}\alpha\in\mathcal{A}(\ominus)\\ d_A\alpha=Z(\bigcirc)\end{matrix}\right\}$$

and we stand a chance to learn something about unknotting numbers algebraically.

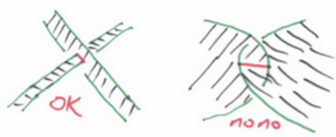
Claim 3

more or less

$$\{\text{Ribbon knots}\}=\left\{u\gamma:\begin{matrix}\gamma\in\mathcal{K}(\bigcirc\!-\!\bigcirc)\\ d\gamma=\bigcirc\bigcirc\end{matrix}\right\}$$

Where:

Ribbon means



and

Algebraic Knot Theory:

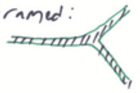
$$\begin{tikzcd} & \mathcal{K}(\bigcirc\bigcirc) \arrow[rr, "Z"] & & \mathcal{A}(\bigcirc\bigcirc)\ni\bigcirc\bigcirc \\ \mathcal{K}(\bigcirc\!-\!\bigcirc) \arrow[ur, "d"] \arrow[rr, "Z"] \arrow[dr, "u"'] & & \mathcal{A}(\bigcirc\!-\!\bigcirc) \arrow[ur, "d"] \arrow[dr, "u"] & \\ & \mathcal{K}(\bigcirc\!\!-\!\!\bigcirc) \arrow[rr, "Z"] & & \mathcal{A}(\bigcirc)\end{tikzcd}$$

So

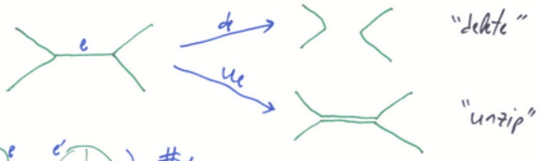
$$Z\left(\left\{\begin{matrix}\text{Ribbon}\\ \text{knots}\end{matrix}\right\}\right)\subset\{u\alpha:d\alpha=Z(\bigcirc\bigcirc)\}\subset\mathcal{A}(\bigcirc\!-\!\bigcirc)$$

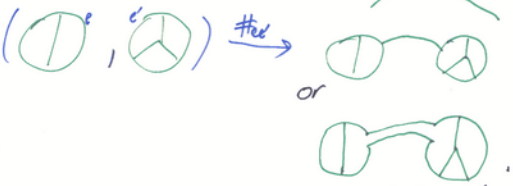
And we stand a chance to find a counterexample to

$$\{\text{Ribbon}\}=\{\text{Slice}\}\;!$$

So many interesting properties of knots are definable using Knotted Trivalent Graphs (KTGs)
(Fully labeled, oriented, Framed: )

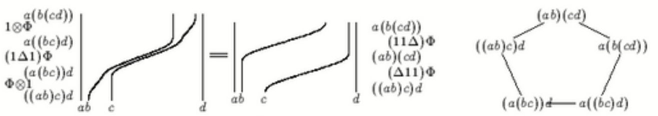
and the basic operations between them:



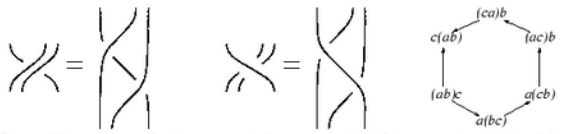


We seek a "TG-morphism" into algebra:

1. $\forall\Gamma$ an algebraic space $A(\Gamma)$, $Z_\Gamma: K(\Gamma)\to A(\Gamma)$.
2. $d, u, \#$ defined on the $A(\Gamma)$'s.
3.
$$\begin{tikzcd} K(\Gamma) \arrow[r, "Z"] \arrow[d, "u_e"] & A(\Gamma) \arrow[d, "u_e"] \\ K(u_e\Gamma) \arrow[r, "Z"] & A(u_e\Gamma) \end{tikzcd}$$
etc.



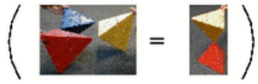
The pentagon relation ⬠ and its tensor-category-theoretical origin.



The positive and negative hexagon relations $\bigcirc_\pm$ and their tensor-categorical origin.

Aside 2

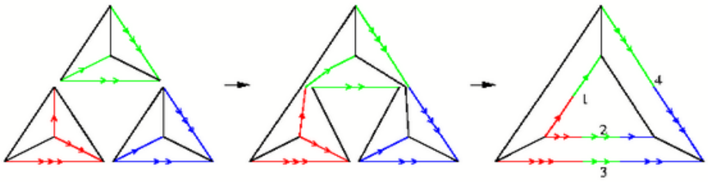
Lest you think it is easy...

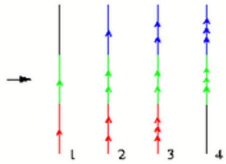


Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi Hopf algebras.

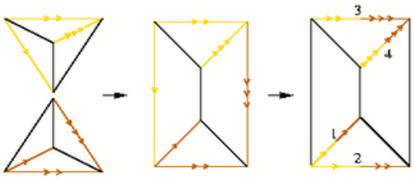
Proof.

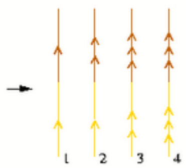
 $:= \Phi \in \mathcal{A}(\uparrow_3)$





$= (\Phi\otimes 1)\cdot(1\otimes\Delta\otimes 1)(\Phi)\cdot(1\otimes\Phi) \in \mathcal{A}(\uparrow_4)$





$= (\Delta\otimes 1\otimes 1)(\Phi)\cdot(1\otimes 1\otimes\Delta)(\Phi) \in \mathcal{A}(\uparrow_4)$