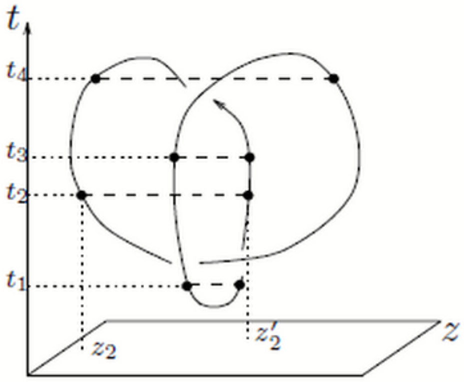
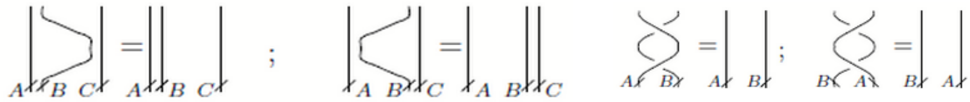
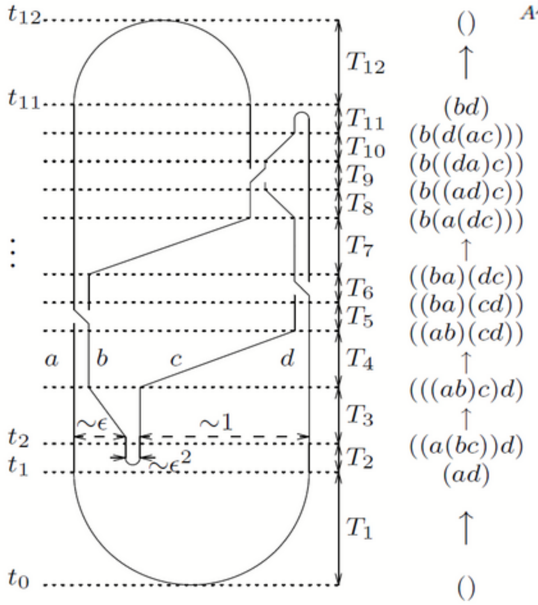
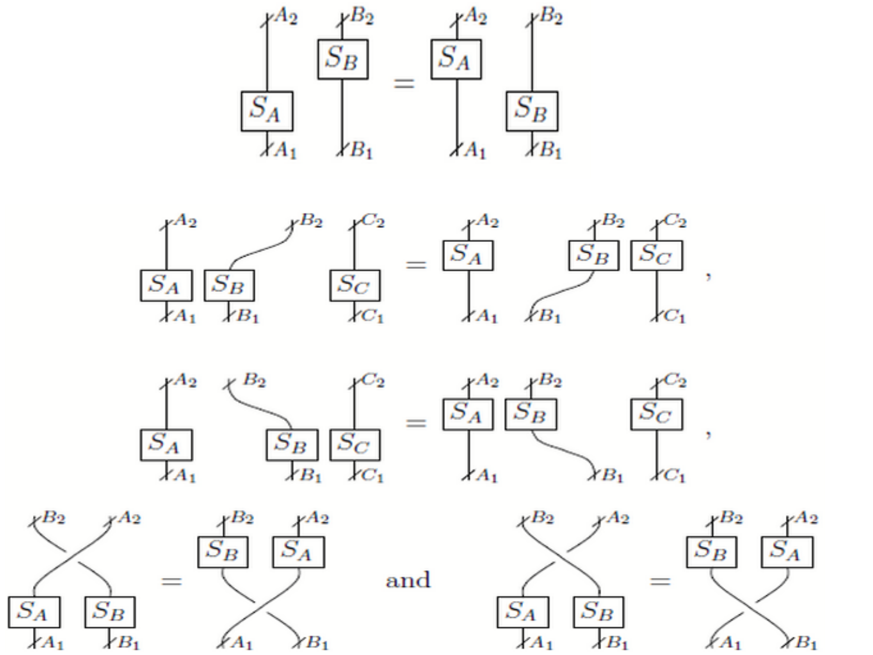


$$(K)=\sum_{m=0}^{\infty}\frac{1}{(2\pi i)^m}\int_{t_{\min}<t_1<\dots<t_m<t_{\max}}\sum_{\substack{\text{applicable}\\ \text{pairings}\\ P=\{(z_i,z_i')\}}}(-1)^{\#P_\downarrow}D_P\bigwedge_{i=1}^{m}\frac{dz_i-dz_i'}{z_i-z_i'}\in\mathcal{A}^r,$$

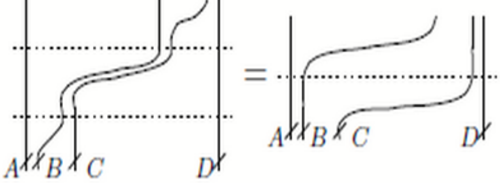
$$\tilde{Z}(K)=Z(K)\,\big/\,(Z(\infty))^{\frac{c}{2}}$$







The pentagon:



$$\longrightarrow\int_{0\le t_1\le\dots\le t_6\le 1}\frac{d(1-t_1)}{1-t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\frac{d(1-t_4)}{1-t_4}\frac{d(1-t_5)}{1-t_5}\frac{dt_6}{t_6}$$

$$=-\int_0^1\frac{dt_6}{t_6}\int_0^{t_6}\frac{dt_5}{1-t_5}\int_0^{t_5}\frac{dt_4}{1-t_4}\int_0^{t_4}\frac{dt_3}{t_3}\int_0^{t_3}\frac{dt_2}{t_2}\int_0^{t_2}\frac{dt_1}{1-t_1}$$

$$\sum_{k_1>0}t_1^{k_1-1}$$

$$\sum_{k_1>0}\frac{t_2^{k_1}}{k_1}$$

$$\sum_{k_1>0}\frac{t_3^{k_1}}{k_1^2}$$

$$\sum_{k_1>0}\frac{t_4^{k_1}}{k_1^3}$$

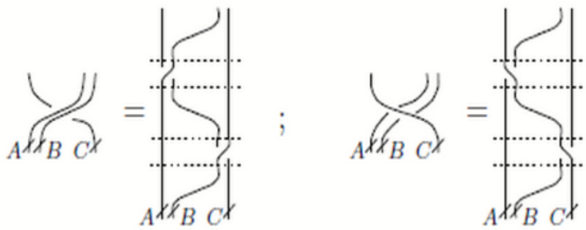
$$\sum_{k_1,k_2>0}\frac{t_5^{k_1+k_2}}{k_1^3(k_1+k_2)}$$

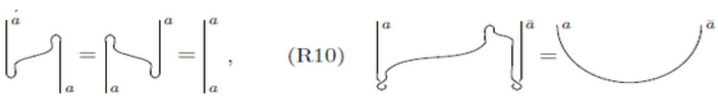
$$\sum_{k_1,k_2,k_3>0}\frac{t_6^{k_1+k_2+k_3}}{k_1^3(k_1+k_2)(k_1+k_2+k_3)}$$

$$=-\sum_{k_1,k_2,k_3>0}\frac{1^{k_1+k_2+k_3}}{k_1^3(k_1+k_2)(k_1+k_2+k_3)^2}$$

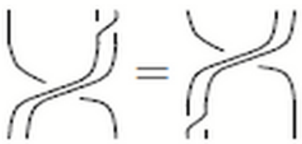
$$=-\sum_{0<n_1<n_2<n_3}\frac{1}{n_1^3n_2n_3^2}=:-\zeta(3,1,2).$$

The hexagons:





R3:



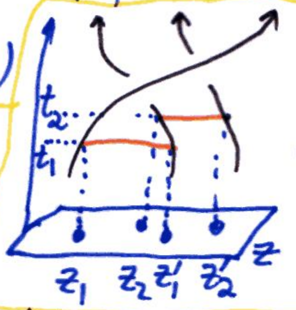
Math 1352 Algebraic Knot Theory – The Knizhnik-Zamolodchikov Connection

Theorem 1. The following is an invariant of braids in $\mathbb{R}_t\times\mathbb{C}_z$ (Fixed endpoints)

$$Z(B)=\sum_m \int_{t_1<\dots<t_m} \frac{D_P}{(2\pi i)^m}\bigwedge_{i=1}^{m} \frac{dz_i-dz_i'}{z_i-z_i'}$$

$P=(\{z_i,z_i'\})$

in $\mathcal{A}(\uparrow_n):=\left\langle t^{ij}: 1\le i\neq j\le n\right\rangle \Big/ \begin{array}{l} t^{ij}=t^{ji}, \\ [t^{ij},t^{kl}]=0 \\ [t^{ij},t^{ik}+t^{jk}]=0 \end{array}$



$\mapsto$

$= \Big\langle$  $\Big\rangle / 4T$

horizontal chords.

= "chord diagrams for braids".

Formal Connections & Curvature.

Let $\Omega\in\Omega^1(M,\overset{\text{graded unital.}}{A})$ with $\deg\Omega=1$.

$\gamma:[0,1]=I\to M$ induces

$\phi:\Delta^m=\{0\le t_1\le\dots\le t_m\le 1\}\to M^m$.

Set $hol_\gamma(\Omega)=P\exp\int_\gamma\Omega=\sum_m\int_{\Delta^m}\phi^*\Omega^m$

where $\Omega^m:=\pi_1^*\Omega\wedge\dots\wedge\pi_m^*\Omega$

Theorem 2. If $F_\Omega:=d\Omega+\Omega\wedge\Omega=0$, then $hol_\gamma(\Omega)$ is invariant under end-point preserving homotopies of γ.

Proof 2. Let $\Gamma: I_s\times I_t\to M$, $\Phi: I_s\times\Delta^m\to M^m$; By Stokes',

$$\int_{\Delta^m}\Phi_1^*\Omega^m-\int_{\Delta^m}\Phi_0^*\Omega^m=\int_{I\times\Delta^m}d\Phi^*\Omega^m-\int_{I\times\partial\Delta^m}\Phi^*\Omega^m=:A_m-B_m$$

Now

$$A_m=\sum_{k=1}^{m}(-1)^{k+1}\int_{I\times\Delta^m}\pi_1^*\Omega\wedge\dots\wedge\pi_k^*d\Omega\wedge\dots\wedge\pi_m^*\Omega$$

and (Fixed endpoints)

$$B_m=\overbrace{\int_{I\times[t_1=0]}\Phi^*\Omega^m}^{=0}\pm\overbrace{\int_{I\times[t_m=1]}\Phi^*\Omega^m}^{=0}+\sum_{k=1}^{m-1}(-1)^k\int_{I\times[t_k=t_{k+1}]}\Phi^*\Omega^m$$

$$=\sum_{k=1}^{m-1}(-1)^k\int_{I\times\Delta^{m-1}}\pi_1^*\Omega\wedge\dots\wedge\pi_k^*(\Omega\wedge\Omega)\wedge\dots\wedge\pi_{m-1}^*\Omega$$

and now $\sum A_m=\sum B_m$ by telescopic summation & $F_\Omega=0$.

The KZ connection.

$M=\mathbb{C}^n\setminus\{\text{diagonals}\}$, $A=\mathcal{A}(\uparrow_n)$,

and $\Omega=\sum_{i<j}t^{ij}w_{ij}$ where $w_{ij}=\frac{dz_i-dz_j}{z_i-z_j}\underset{\text{locally}}{=}d\log(z_i-z_j)$

Compute $F_\Omega=d\Omega+\Omega\wedge\Omega$: $dw_{ij}=0$ so $d\Omega=0$.

$$\Omega\wedge\Omega=\sum_{\substack{i<j\\ k<l}}t^{ij}t^{kl}w_{ij}\wedge w_{kl}=\underbrace{A}_{|\{ijkl\}|=2}+\underbrace{B}_{=3}+\underbrace{C}_{=4}$$ where

$A=C=0$ as $[t^{ij},t^{kl}]=0$ if $|\{ijkl\}|=2$ or 4 and

$$B=\sum_{\alpha<\beta<\gamma}[t^{\alpha\beta},t^{\beta\gamma}]w_{\alpha\beta}\wedge w_{\beta\gamma}+\text{cyclic perms}$$

$$=\sum_{\alpha<\beta<\gamma}\Upsilon^{\alpha\beta\gamma}\big(\underbrace{w_{\alpha\beta}\wedge w_{\beta\gamma}+\text{cyc perms}}_{=0\text{ by "Arnold's identity"}}\big)=0$$

Note: by 4T, $[t^{\alpha\beta},t^{\beta\gamma}]=\Upsilon^{\alpha\beta\gamma}=$ 

Proof of 1

Simply take in theorem 2, γ = the braid and Ω = the KZ connection.

Dror Bar-Natan, Feb 13, 2007