267ODEs-121130, Hour 33: Changing the independent variable, Sturm comparison

November-28-12
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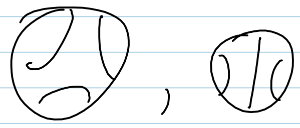
Riddle Along. $\#\Big($  $, \ldots \Big) = ?$

Reminder: $y'' + qy = 0$,

$q < 0$: at most one zero

$q > 0$, $\int_A^\infty q\,dt = \infty$: oscillates.

In the other direction, we have the following:

Theorem 3.3. *Let $A > 0$ be given. If $q(x)$ is continuous and $q(x) > 0$ for all $x \geq A$ and if $\int_A^\infty xq(x)dx < \infty$, and if y is a solution of $y'' + qy = 0$, then*

(1) *There is some $B > A$ beyond which y has no zeros.*
(2) *There is a constant K such that*

$$\lim_{x\to\infty} y'(x) = K = \lim_{x\to\infty} \frac{y(x)}{x}$$

Comment 3.3. I could not prove or find a counterexample to the statement that above, K is always non-zero. If this is true then the first statement above is superfluous as it would immediately follow from the second. I didn't have time to consult with the references, [CL, page 103, problem 28] and [Co, page 92 Theorem 3].

Proof. Find $C > A$ such that $\int_C^\infty xq\,dx < 1$, and assume that y has at least two zeros beyond C; let a be the first of those and let b be the second. Let $\alpha = y'(a)$; without loss of generality we may assume that $\alpha > 0$. Then $y'(b) < 0$ and by convexity we have that on $[a, b]$, $y(x) \leq \alpha(x - a) < \alpha x$. So

$$\alpha \leq y'(a) - y'(b) = -\int_a^b y''(x)dx = \int_a^b yq\,dx \leq \int_a^b \alpha xq\,dx \leq \alpha \int_C^\infty xq\,dx < \alpha,$$

a contradiction. Therefore y cannot have two further zeros beyond B, and (1) is proven.

Now we know that beyond some point D, y is non-zero. Without loss of generality it is positive and therefore convex. It therefore lies below any of its tangents, and therefore on $[D, \infty]$ it is bounded by some linear function βx. Hence for any $a < b$ in $[D, \infty]$,

$$|y'(a) - y'(b)| = \left|\int_a^b y''dx\right| = \int_a^b yq\,dx \leq \int_a^b \beta xq\,dx \leq \beta \int_a^\infty xq\,dx,$$

and the last integral goes to 0 when $a \to \infty$. Hence $y'(x)$ is a "Cauchy function" (the "function" analog of a "Cauchy sequence"), and hence it converges to some limit K. The rest follows from L'Hôpital. □

Exercise 3.5. Show that all solutions of $y'' + x^\alpha y = 0$ are oscillatory for $x > 1$ if $\alpha > -1$. For what value of α does Theorem 3.3 apply to determine the large x behaviour of such solutions?

4.2. **Changing the Independent Variable.** If y satisfies $y'' + p(x)y' + q(x)y = 0$ and we set $z = \nu(x)$, where ν satisfies $\nu'' + p\nu' = 0$, then the equation becomes

$$\frac{d^2y}{dz^2} + Q(z)y = 0, \qquad \text{where} \qquad Q(z) = \frac{q(x(z))}{[\nu'(x(z))]^2}.$$

$$\frac{d^2y}{dz^2} + Q(z)y = 0, \quad \text{where} \quad Q(z) = \frac{1}{[\nu'(x(z))]^2}.$$

The zeros of y get moved by this transformation, so studying the oscillatory behaviour of $y(x)$ as $x \to \infty$ corresponds to studying the oscillatory behaviour of $y(z)$ as $z \to \lim_{x\to\infty} \nu(x)$, and the latter point may or may not be ∞. Note though, that the amplitudes of oscillations (if they occur), are unchanged.

Example 4.2. Under the change of independent variable $z(x) = x^3/3$, the equation $y'' - \frac{2}{x}y' + y = 0$ becomes the equation $\frac{d^2y}{dz^2} + \frac{1}{(3z)^{4/3}}y = 0$:

Indeed, $\nu'' - \frac{2}{x}\nu' = 0 \Rightarrow \nu' = x^2 \Rightarrow z = \nu = x^3/3$,

so $x = \sqrt[3]{3z}$ so $Q = \frac{1}{((\sqrt[3]{3z})^2)^2} = \frac{1}{(3z)^{4/3}}$

Skipped

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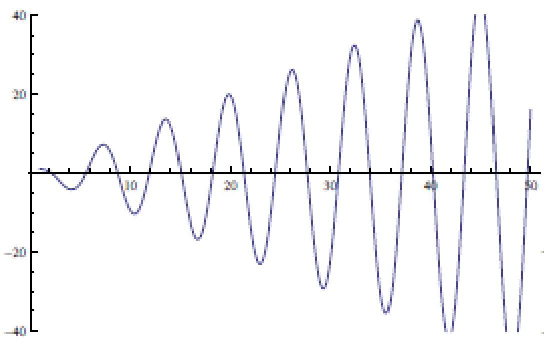
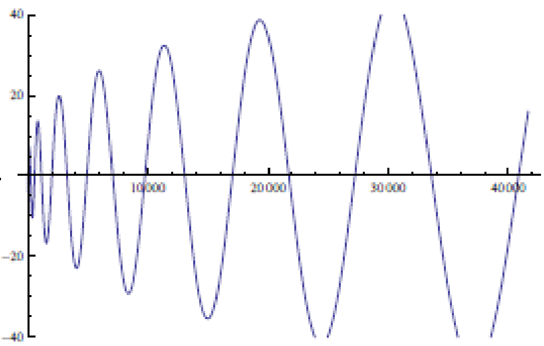
a = 50; b = 40;
ψ = NDSolve[
   y''[x] - 2/x y'[x] + y[x] == 0
    && y[1] == 1 && y'[1] == 0,
   y[x], {x, 1, a}
  ];
Plot[Evaluate[y[x] /. ψ],
 {x, 1, a}, PlotRange → {-b, b}]
```

```
ψ = NDSolve[
   y''[z] + 1/(3 z)^(4/3) y[z] == 0
    && y[1/3] == 1 && y'[1/3] == 0,
   y[z], {z, 1, a^3/3}
  ];
Plot[Evaluate[y[z] /. ψ],
 {z, 1, a^3/3}, PlotRange → {-b, b}]
```





Charles Sturm, 1803–1855

Theorem 5.1. *(The Sturm Comparison Theorem) Suppose y_1 satisfies $y_1'' + q_1y_1 = 0$ and y_2 satisfies $y_2'' + q_2y_2 = 0$ and suppose $q_2 > q_1$ in some interval. Then in the open interval between any two zeros of y_1 there is a zero of y_2 (hence y_2 oscillates more rapidly than y_1).*

Proof. Consider $W(x) := y_1(x)y_2'(x) - y_2(x)y_1'(x)$. Then

$$W' = y_1y_2'' - y_2y_1'' = (q_1 - q_2)y_1y_2.$$

Now argue by contradiction. Suppose $y_1(a) = 0 = y_1(b)$ and $y_1 > 0$ on (a, b). Suppose also that y_2 has no zeros on (a, b); without loss of generality, $y_2 > 0$ on (a, b). Then by the above equality and by $q_1 < q_2$, it follows that W is decreasing on (a, b). Yet $W(a) = -y_2(a)y_1'(a) \leq 0$ and $W(b) = -y_2(b)y_1'(b) \geq 0$. □

Did comparison $y'' + x^2y = 0$ with $y'' + x^{-2}y = 0$

Corollary 5.1. Assuming $y'' + qy = 0$, if q is increasing the the distance between successive zeros of y is decreasing, and if q is decreasing then the distance between successive zeros of y is increasing.

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not done.

Example 5.2. As we have seen in Example 4.1 the Bessel equation of order 0 is equivalent to the equation $V'' + \left(1 + \frac{1}{4x^2}\right) V = 0$. Hence the distance between successive zeros of the Bessel equation of order 0 is increasing and by comparison with $v'' + v = 0$, it converges to π:

```
zs = x /. Table[FindRoot[y[x] /. J0, {x, λ}], {λ, 2.8, 50, 3.14}]
```

```
{2.91009, 6.03123, 9.16593, 12.3041, 15.4436, 18.5839, 21.7245, 24.8654,
 28.0064, 31.1475, 34.2888, 37.43, 40.5714, 43.7127, 46.8541, 49.9956}
```

```
Table[zs[[j + 1]] - zs[[j]], {j, 1, 15}]
```

```
{3.12114, 3.1347, 3.13816, 3.13954, 3.14023, 3.14062, 3.14087,
 3.14103, 3.14114, 3.14123, 3.14129, 3.14133, 3.14137, 3.1414, 3.14143}
```