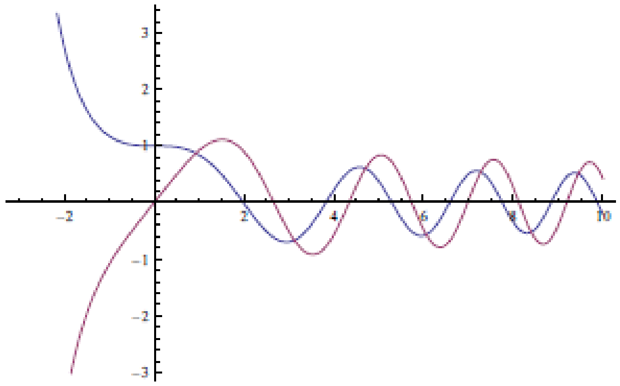
November-26-12
8:18 PM

HW9 on web by midnight?
Final bits of RSP @tutorial today!

```
Ai1 = NDSolve[y''[x] + x y[x] == 0 && y[0] == 1 && y'[0] == 0,
   y[x], {x, -3, 10}];
Ai2 = NDSolve[y''[x] + x y[x] == 0 && y[0] == 0 && y'[0] == 1,
   y[x], {x, -3, 10}];
Ai = Join[Ai1, Ai2]

{{y[x] → InterpolatingFunction[{{-3., 10.}}, <>][x]},
 {y[x] → InterpolatingFunction[{{-3., 10.}}, <>][x]}}
Plot[Evaluate[y[x] /. Ai], {x, -3, 10}]
```



on board

Added Dec 5, 2012: I should have centered the qualitative analysis discussion around one or two examples & study them to the ultimate detail.

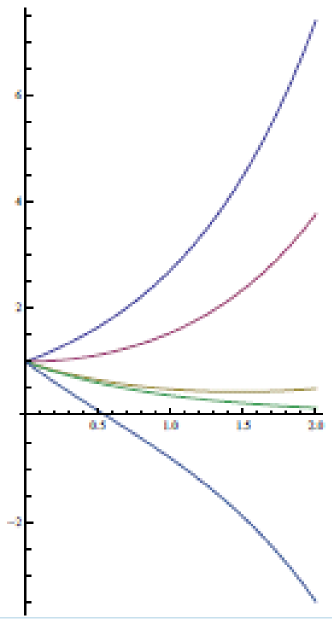
Some general words about $y''+qy=0$ & "returning forces"

Theorem 3.1. *If $q(x) < 0$ for every x in some connected subset I of $\mathbb{R}$, then any solution of $y'' + qy = 0$ may have at most one zero on I.*

Example 3.1. Consider the solutions of $y'' - y = 0$ with $y(0) = 1$ and $y'(0) = c$, for $c \in \{1, 0, -0.9, -1, -2\}$.

"too common sensy to be given a formal proof".

```
Plot[Evaluate[Table[
   y[x] /.
    DSolve[y''[x] - y[x] == 0
      && y[0] == 1 && y'[0] == c,
     y[x], x],
   {c, {1, 0, -0.9, -1, -2}}
  ]], {x, 0, 2}, AspectRatio → 2]
```



Example. Airy $y''+xy=0$ & negative x.

Theorem 3.2. *If $q(x)$ is continuous and $q(x) > 0$ for all $x \geq A$ and if $\int_A^\infty q(x)dx = \infty$, then any solution to $y'' + qy = 0$ has infinitely many zeros for $x \geq A$.*

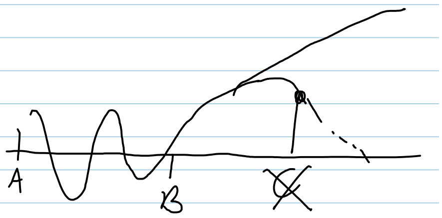
Proof. Suppose not. Then there is a solution y for which $y(x) > 0$ for all $x \geq B$, for some $B \geq A$. If we had $y'(C) \leq 0$ for some $C > B$, then as $y'' < 0$ and therefore y' is decreasing, we'd have that $y'(x) < 0$ for all $x > C$, and therefore there is some $x > C$ with $y(x) = 0$. So it must be that $y'(x) > 0$ for all $x \geq B$. Now consider $V(x) := -\frac{y'(x)}{y(x)}$. We already know it is negative for all $x \geq B$. Yet

$$V' = -\frac{y''y - y'^2}{y^2} = \frac{qy^2 + y'^2}{y^2} = q + V^2,$$

and hence

$$V(x) = V(B) + \int_B^x V'(t)dt = V(B) + \int_B^x V^2 dt + \int_B^x q dt.$$

But as $\int_B^\infty q(t)dt$ is divergent, the above quantity will become positive for large enough x, contradicting the negativity of $V(x)$. ☐



$V(x) = -\frac{y'}{y} < 0$

yet

$V' = V^2 + q$

☐

Example: Airy @ positive x.

But what about Bessel

$x^2 y'' + xy' + (x^2 - \nu^2) y = 0$?

4.1. **Changing the Dependent Variable.** If y satisfies $y'' + p(x)y' + q(x)y = 0$ and we set $y = \mu(x)V$, where μ satisfies $2\mu' + p\mu = 0$, then V satisfies $V'' + Q(x)V = 0$, where $Q = q - \frac{1}{4}p^2 - \frac{1}{2}p'$. The good news is that V has exactly the same zeros as y, so the "frequency" of the oscillatory behaviour of y may be studied by studying $V'' + Q(x)V = 0$. Though note that "amplitudes" are modified.

Example 4.1. For Bessel's equation of order 0, $y'' + \frac{1}{x}y' + y = 0$, which appeared here in Example 1.1, setting $V = \sqrt{x}y$ yields the equation $V'' + \left(1 + \frac{1}{4x^2}\right)V = 0$, which oscillates by Theorem 3.2:

All done, except the $\sqrt{x}$ factor was not computed.

```
{V0} = NDSolve[
   V''[x] + (1 + 1/(4 x^2)) V[x] == 0
    && V[1] == 1 && V'[1] == 1/2,
   V[x], {x, 1, 50}
  ];
Plot[Evaluate[{y[x] /. J0, V[x] /. V0}], {x, 1, 50}]
```