Today's Topics.
1. $xy'+p(x)y=0$ Why? Why little? How? Qualitative behaviour?
2. $x^2y''+xpy'+qy=0$ Why? why this? How? Qualitative behaviour?

Topic 1. $xy'+p(x)y=0$ p analytic [has a convergent P.S. expansion]

Why? [First example of singularity]

Why not $r(x)y'+p(x)y=0$? $/\cdot\frac{x}{r(x)}$

get $xy'+x\frac{p}{r}y=0 \Rightarrow$ (we are studying any such eqn in which xp/r is analytic)

Why little: We could have solved it explicitly anyway.

How? $xy'+p_0y=0 \rightsquigarrow y=x^{-p_0}=x^\alpha$

So try $x^\alpha\sum a_nx^n=\sum a_nx^{n+\alpha}$ $[a_0=1]$

in $xy'+py=0$; get $(n+\alpha)a_n+\sum_{j=0}^{n}p_j\cdot a_{n-j}=0$

$$(n+\alpha+p_0)a_n=-\sum_{j=1}^{n}p_ja_{n-j}$$

@ $n=0$: $\alpha=-p_0$; otherwise all is well & we even have Fuchs' Theorem.

Qualitative behaviour:

y looks like x^α, plus corrections.

example: $xy'+(e^x+e^{-x})y=0$ diverges like

$x^{-\alpha}$ near $x=0$.

Topic 2: $xy'+py=0 \to x^2y''+pxy'+qy=0$ w/ p,q analytic

why not $xy''+py'+qy=0$? [That's a special case]

why not $x^2y''+py'+qy=0$? [Too hard; can pretend to solve by P.S., but P.S. ALMOST NEVER converges; other techniques later]

what about $ry''+py'+qy=0$? $/\cdot \frac{x^2}{r}$

$\rightsquigarrow$ need $\frac{xp}{r}$, $\frac{x^2q}{r}$ to be analytic.

How? Try x^α in $x^2y''+p_0xy'+q_0y=0$

$\Rightarrow \alpha$ must be a root of $F(\alpha)=\alpha(\alpha-1)+p_0\alpha+q_0=0$

Now try $x^\alpha\sum a_nx^n=\sum a_nx^{n+\alpha}$ $(a_0=1)$.

Get

$$F(\alpha+n)\,a_n=-\sum_{k=0}^{n-1}a_k\left[(\alpha+k)p_{n-k}+q_{n-k}\right]$$

$$y_1=x^{r_1}\left(1+\sum_{n=1}^{\infty}a_nx^n\right)$$

$$y_2=\begin{cases} y_1\log x+x^r\sum_{n=1}^{\infty}b_nx^n & r_1=r_2=r\\ cy_1\log x+x^{r_2}\left(1+\sum_{n=1}^{\infty}b_nx^n\right) & r_1-r_2=N\in\mathbb{N}_{>0}\\ x^{r_2}\left(1+\sum_{n=1}^{\infty}b_nx^n\right) & \text{otherwise.}\end{cases}$$

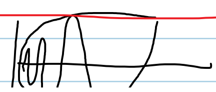
skipped.

There is always a Fuchs' theorem!

Example 1. $y''+\left(\frac{1}{2x^2}+\frac{1}{2(1-x^2)}\right)y=0$

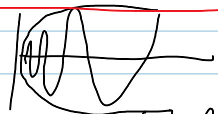
$F(\alpha)=\alpha(\alpha-1)+\frac{1}{2}=0$ $\quad\alpha=\frac{1}{2}\pm\frac{1}{2}i$,

done line

Sol'ns look like 

done

lix

Sol'ns look like 

$$\sqrt{x}\left(A\sin\left(\tfrac{1}{2}\log x\right)+B\cos\left(\tfrac{1}{2}\log x\right)\right)$$

Example 2. Bessel's equation of order λ:

$$y''+\frac{1}{x}y'+\left(1-\frac{\lambda^2}{x^2}\right)y=0 \qquad F(\alpha)=\alpha^2-\lambda^2$$

$$\alpha_1=+\lambda \qquad \alpha_2=-\lambda$$

2λ not an integer: Sol'n behave mostly like $x^{-\lambda}$, exceptionally like x^{λ}

$\lambda=0$ Sol'n behave mostly like $\log x$, exceptionally like 1.

2λ a positive integer – Same as before.