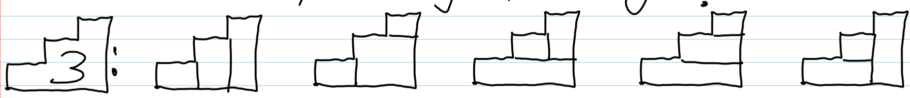
HW7 is on web.

Riddle Along. How many ways are there to tile the n-staircase w/ exactly n rectangles?



$$y''+p(x)y'+q(x)y=0 \quad \frac{v_1=y}{v_2=y'}, \quad v'=\begin{pmatrix}0 & 1\\ -q & -p\end{pmatrix}v$$

y_1, y_2 indep. sol'ns, $W=\det\begin{pmatrix}y_1 & y_2\\ y_1' & y_2'\end{pmatrix}$

$W'=-pW$ ("Abel's Thm")

Example. $y''+y=0$

Problem Find a power series $y(x)=\sum_{k=0}^{\infty} a_k(x-x_0)^k$ that solves the equation $y'=f(x,y)$

Motivation: 1. Minor: Combinatorics.

2. Major: Power series are lousy at large scale, but excellent at small scales. See QED w/ $\alpha=\frac{1}{137}$.

```
In[1]:= PowerSeriesSolve[f_, x0_, y0_, n_] := (
          ϕ[0] = y0;
          Do[
           ϕ[k] = y0 + ∫_x0^x (Normal[Series[f /. y → ϕ[k - 1], {x, x0, k - 1}]] /. x → t) ⅆt,
           {k, 1, n}
          ];
          ϕ[n]
        );

In[2]:= PowerSeriesSolve[Sqrt[1 + y^2], 0, 0, 5]
```

Out[2]= $x+\frac{x^3}{6}+\frac{x^5}{120}$

```
In[3]:= PowerSeriesSolve2[f_, x0_, y0_, n_] := Module[{ϕ = y0},
          Do[
           ϕ += (D[f /. y → ϕ, {x, k - 1}] /. x → x0)/k! x^k,
           {k, 1, n}
          ];
          ϕ
        ];
```

$\phi'=f(x,\phi) \quad \phi(x_0)=y_0 \quad \phi=\sum_{k=0}^{\infty}a_k(x-x_0)^k$

Differentiate both sides $(k-1)$ times, evaluate at x_0:

$$k!\,a_k=\frac{d^{k-1}}{dx^{k-1}}f(x,\phi(x))\Big|_{x=x_0}=$$

$$=\frac{d^{k-1}}{dx^{k-1}}f\Big(x,\underbrace{\sum_{j=0}^{k-1}a_j(x-x_0)^j}_{\phi_{k-1}}\Big)\Big|_{x=x_0}$$

$$\phi_k=\phi_{k-1}+\frac{x^k}{k!}\frac{d^{k-1}}{dx^{k-1}}f(x,\phi_{k-1})\Big|_{x=x_0}$$

```
In[4]:= PowerSeriesSolve2[Sqrt[1 + y^2], 0, 0, 10]
```

Out[4]= $x+\frac{x^3}{6}+\frac{x^5}{120}+\frac{x^7}{5040}+\frac{x^9}{362880}$

```
In[5]:= Series[Sinh[x], {x, 0, 10}]
```

Out[5]= $x+\frac{x^3}{6}+\frac{x^5}{120}+\frac{x^7}{5040}+\frac{x^9}{362880}+O[x]^{11}$

Little on power series.

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Thm 1 Given $\sum a_n x^n$ $\exists R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, "radius of convergence", s.t. $\sum a_n x^n$ absolutely converges if $|x|<R$ & diverges if $|x|>R$.
[if $|x|=R$, "it depends"].

PF $R = \sup\{r : a_n r^n \xrightarrow[n\to\infty]{} 0\}$

$= \sup\{r : |a_n r^n| \text{ is bndd}\}$

done, fine

Thm 2 (loose) 1. If f has a formula, it has a natural extension to $\mathbb{C}$.

2. In that case, R is the distance from 0 to the nearest point in $\mathbb{C}$ in which the formula fails.

Examples. 1. $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

2. $\frac{1}{1+x^2} = 1-x^2+x^4-x^6 \dots$

Problem Given $y''+p(x)y'+q(x)y=g(x)$, find a power series $y=\sum_{n=0}^{\infty} a_n x^n$ that solves this eqn.

Do the $y''+y=0$ example.

Do the Airy example $y''=xy$

State Fuchs' Theorem: The series for $y(x)$ converges at radius at least the least of the radii for p, q, g.

Pensieve header: Plotting the solutions of the Airy equation, $y''=xy$.

```
In[1]:= Ai1 = NDSolve[y''[x] == x y[x] && y[0] == 1 && y'[0] == 0, y[x], {x, -10, 3}];
        Ai2 = NDSolve[y''[x] == x y[x] && y[0] == 0 && y'[0] == 1, y[x], {x, -10, 3}];
        Ai = Join[Ai1, Ai2]
Out[3]= {{y[x] → InterpolatingFunction[{{-10., 3.}}, <>][x]},
         {y[x] → InterpolatingFunction[{{-10., 3.}}, <>][x]}}
In[4]:= Plot[Evaluate[y[x] /. Ai], {x, -10, 3}]
Out[4]=
```