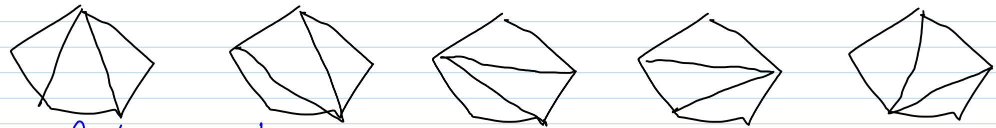
Riddle Along. By marking diagonals, how many ways are there to divide an n-gon into triangles?

$n=5$:



Claim 1 Solutions to $V'(t)=A(t)V(t)$ exist and are unique wherever $A(t)$ is continuous.

Claim 2 If $\Psi'(t)=A(t)\Psi(t)$, then either Ψ is regular for all t or singular for all t.

Debt. pf2 using the "Wronskian" & det'.

Proof of Claim 1 We'll prove exis/uniq on $I_0=[-1{,}000, 1{,}000]$.

By compactness, find μ s.t. $|A_{ij}|<\mu$ on I_0.

Take $t_0\in I_0$ and let $R=[t_0-a, t_0+a]\times B(V_0, |V_0|)$, using the largest a that fits.

$|A_{ij}|<\mu$ on I_0 so $A(t)V(t)$ is Lips. w/ $K=n\mu$ on R

so exist/uniq. holds on $[-\delta,\delta]$ where $\delta=\min(a, \frac{|V_0|}{M})$.

Yet $M<n\cdot\mu\cdot 2|V_0|$ so $\frac{|V_0|}{M}>\frac{1}{2\mu n}=\epsilon$, a fixed positive constant indep. of t_0, V_0.

So wherever a sol'n exists, it exists an ϵ further (within I_0).

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pf2 Use the Wronskian $W=\det\Psi(t)$.

Aside what $\det(M(t))'$?

Sol. $|M(t+\epsilon)|\sim|M(t)+\epsilon M'(t)|=$

$=|M(I+\epsilon M^{-1}M')|\sim|M|(1+\epsilon\,\mathrm{tr}(M^{-1}M'))$

so $\det(M(t))'=|M|\,\mathrm{tr}(M^{-1}M')$.

In our case,

$W(t+\epsilon)=\det(\Psi(t+\epsilon))=\det(\Psi(t)+\epsilon\Psi')=\det(\Psi+\epsilon A\Psi)$

$=\det(I+\epsilon A)\det\Psi=(1+\epsilon(\mathrm{tr}A))W$

So $W' = (\mathrm{tr}\,A)\cdot W$ So $W = \exp\int \mathrm{tr}(A)\,dt \cdot W(0)$

Q. What is the corresponding theory for

$$y'' + p(x)y' + q(x)y = 0 \,?$$

Ans.

$$y'' + py' + qy = 0 \xleftrightarrow[V_2 = y']{V_1 = y} \begin{matrix} V_1' = V_2 \\ V_2' = -qV_1 - pV_2 \end{matrix} \iff V' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} V$$

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ always 0 or never 0.

$$W' = -pW \Rightarrow W(t) = \exp\left(-\int_{t_0}^{t} p(s)\,ds\right)\cdot W(t_0)$$

done [illegible]

[Aside: do the case of $y'' + y = 0$]

$$y'' + py' + qy = g \iff V' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} V + \begin{pmatrix} 0 \\ g \end{pmatrix}$$

If $y_{1,2}$ are sol'n of homo, $\Psi = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$

$V = \Psi \int \left(\Psi^{-1}\begin{pmatrix} 0 \\ g \end{pmatrix}\right) dt \ldots$

Problem Given $y'' + p(x)y' + q(x)y = 0$, Find a power series $y = \sum_{n=0}^{\infty} a_n x^n$ that solves this eqn.

Motivation: 1. Minor: Combinatorics.

2. Major: Power series are lousy at large scale, but excellent at small scales. See QED w/ $\alpha = \frac{1}{137}$.

Do the $y'' + y = 0$ example.