267ODEs-121029, Hour 21: Matrix Exponentiation

October-24-12
9:49 AM

Wiki pages and files must begin with 12-267!
HW5 is on web!
TT results hopefully tomorrow.
Read Along: BDP Chapter 7.
Riddle along: the game of 15.

Systems of CC. lin. hom. ODEs $y' = Ay \quad y(0) = y_0$

Sol'n $y(x) = e^{Ax} \cdot y_0$ What's e^{Ax}?

Define $e^{tA} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n$

Properties 1. Converges! [From this point, no worry about convergence]

2. $e^{0A} = I$

3. $e^{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}} = \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}$

4. Satisfies $(e^{tA})' = Ae^{tA}$

5. $e^{A+B} = e^A e^B$ whenever $AB = BA$.

6. $e^{(t+s)A} = e^{tA} e^{sA}$

7. $e^{C^{-1}AC} = C^{-1} e^A C$

done line.

8. (later) $\exp(\lambda I + J) = \ldots$

$\Rightarrow$ Totally computable!

Example Solve

$\dot{x} = 4x - 6y \qquad x(0) = 2$

$\dot{y} = 3x - 5y \qquad y(0) = -1$

In general, diagonalization works at least when the characteristic poly. has n distinct roots

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In[1]:= D1 = (1 0; 0 -2); D2 = (2 1; 0 2); CC = (1 -1; -1 2);
        Inverse[CC] // MatrixForm
Out[2]//MatrixForm=
  (2 1
   1 1)
In[3]:= MatrixForm /@ {D1, CC.D1.Inverse[CC]}
Out[3]= {(1 0; 0 -2), (4 3; -6 -5)}
In[4]:= MatrixForm /@ {D2, CC.D2.Inverse[CC]}
Out[4]= {(2 1; 0 2), (3 1; -1 1)}
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Thm (Jordan canonical form) If $T: V \to V$ is a linear transformation [over $\mathbb{C}$], then

There is a basis $\beta = (v_1, \dots, v_n)$ of V s.t.

$$D = [T]_\beta = \begin{pmatrix} \boxed{B_1} & 0 & 0 \\ 0 & \boxed{B_2} & 0 \\ 0 & 0 & \boxed{B_3} \end{pmatrix} \qquad B_i = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

Exponentiate $(\lambda I + J)$

Example solve

$$\dot{x} = 3x - y \qquad x(0) = 2$$
$$\dot{y} = x + y \qquad y(0) = -1$$