Read Along. BDP chapter 8.
Riddle Along. Can you find uncountably many sets of integers, the intersection of any two of which is finite?

<u>Thm</u> Given $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$,
if $P_0 \in \mathbb{R}^2$, $g(P_0) = 0$, $\nabla g(P_0) \neq 0$, $\nabla f(P_0) \not\propto \nabla g(P_0)$,
then arbitrarily near P_0 there are points
P_+ & P_- with $g(P_+) = g(P_-) = 0$ and
$$f(P_+) > f(P_0) > f(P_-)$$

$\Longleftrightarrow \nexists \lambda \;\; \nabla(f + \lambda g) = 0$

<u>Pf</u> Find $V \perp \nabla g(P_0)$ with $V \cdot \nabla f(P_0) \neq 0$. Then pick $P_\pm$ very near $P_0 \pm \epsilon V$

In CoV,
$$f \rightsquigarrow \int F(x, y, y')\,dx = J(y)$$
$$g \rightsquigarrow \int G(x, y, y')\,dx = K(y)$$
and I only wish to argue that $\nabla_p \rightsquigarrow EL_\phi$
where $EL_\phi(F) := F_y - \frac{d}{dx} F_{y'}\Big|_{\text{at } \phi}$.

Indeed $D_v f = \frac{d}{d\epsilon} f(p + \epsilon v)\Big|_{\epsilon=0} = (\nabla f, v)$
$$D_h J = \frac{d}{d\epsilon} J(\phi + \epsilon h)\Big|_{\epsilon=0} = \int \left(F_y - \frac{d}{dx} F_{y'}\right) h\,dx$$
$$= (EL_\phi(F), h).$$

שיטות נומריות:

1. שימוש במשפט פיקרד: אין נומרי! דוגמה $\begin{cases} y' = -y \\ y(0) = 1 \end{cases}$

done line

2. שיטת אוילר

$x_0 \quad x_{n+1} = x_n + h$

$y_0 \quad y_{n+1} = y_n + h y_n' \; (= y_n + h f(x_n, y_n))$

דוגמה - תרגיל $y(1)$ על $h = 1, \frac{1}{2}, \frac{1}{3}$

$y(1) = 0.3679$

$h=1 \to 0$

$h=\frac{1}{2} \to \frac{1}{4} = 0.25$

$h = \frac{1}{3} \to 8/27 \approx 0.2963$

כמה מהר מגיעים לתוצאה?

שגיאה מקומית: (מניחים y_n מדויק)

$$\phi(x_n+h) = \phi(x_n) + h\phi'(x_n) + O(h^2)$$

$$y_{n+1} = y_n + h y_n' \qquad O(h^2)$$

שגיאה מצטברת:

תשובה מאוד: $E \sim \frac{x-x_0}{h} \cdot h^2 \sim h$

שיקולי "רגיעת דרגות" משמעותיים יותר התוצאה.

מסקנה - כדי להגיע לדיוק h 10^{-6} צריך 10^{9} חישובים - וזה לא אפשרי.

עד $x \sim 1{,}000$

3. שיטת אוילר משופרת: $x_{n+1} = x_n + h$,

$$\phi(x_{n+1}) = \phi(x_n) + \int_{x_n}^{x_{n+1}} f(x, \phi(x))\,dx$$

$$y_{n+1} = y_n + \frac{f(x_n,y_n) + f(x_{n+1}, y_{n+1})}{2} h = y_n + \frac{y_n' + f(x_n+h, y_n + h y_n')}{2} h$$

בצורה אחרת: נתונים (x_n, y_n)

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + h, y_n + k_1 h)$$

$$y_{n+1} = y_n + \frac{k_1 + k_2}{2} h$$

הערכת שגיאה: מקומית $\sim h^3$

מצטברת $\sim h^2$ לדיוק 10^{-6} צריך $2 \cdot 10^3$,

4. שיטה (כללית) משפחת שיטות:

$k_1 = f(x_n, y_n)$

$k_2 = f(x_n + \alpha_2 h, y_n + \alpha_2 k_1 h)$

$k_3 = f(x_n + \alpha_3 h, y_n + \alpha_3 k_2 h)$

$k_7 = \ldots$

$y_{n+1} = y_n + (\beta_1 k_1 + \ldots + \beta_7 k_7) h$

ולבחור α_i, β_i כך שהיה מקסימלי

5. שיטת Runge-Kutta

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \text{כש} \quad \begin{cases} k_1 = f(x_n, y_n) \\ k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}h k_1) \\ k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}h k_2) \\ k_4 = f(x_n + h, y_n + h k_3) \end{cases}$$

מקומית $\sim h^5$

מצטבר $\sim h^4$

From LocalRungeKutta-2.nb:

In[1]:= $$\phi[x_0] = y_0;$$
$$\phi_0[x\_] := \phi[x];$$
$$\phi_{k\_}[x\_] \;/;\; k \geq 1 := \phi_k[x] = \text{Expand}\big[$$
$$\quad \partial_x(\phi_{k-1}[x]) \;/.\; \phi_0'[x] \mapsto f[x, \phi_0[x]]$$
$$\big];$$
$$\text{ser1} = \sum_{k=0}^{4} \frac{1}{k!} \phi_k[x]\, h^k \;/.\; x \to x_0$$

Out[4]= $$h f[x_0, y_0] + y_0 + \frac{1}{2} h^2 \left(f[x_0, y_0] f^{(0,1)}[x_0, y_0] + f^{(1,0)}[x_0, y_0]\right) +$$
$$\frac{1}{6} h^3 \Big(f[x_0, y_0] f^{(0,1)}[x_0, y_0]^2 + f[x_0, y_0]^2 f^{(0,2)}[x_0, y_0] +$$
$$f^{(0,1)}[x_0, y_0] f^{(1,0)}[x_0, y_0] + 2 f[x_0, y_0] f^{(1,1)}[x_0, y_0] + f^{(2,0)}[x_0, y_0]\Big) +$$
$$\frac{1}{24} h^4 \Big(f[x_0, y_0] f^{(0,1)}[x_0, y_0]^3 + 4 f[x_0, y_0]^2 f^{(0,1)}[x_0, y_0] f^{(0,2)}[x_0, y_0] +$$
$$f[x_0, y_0]^3 f^{(0,3)}[x_0, y_0] + f^{(0,1)}[x_0, y_0]^2 f^{(1,0)}[x_0, y_0] +$$
$$3 f[x_0, y_0] f^{(0,2)}[x_0, y_0] f^{(1,0)}[x_0, y_0] + 5 f[x_0, y_0] f^{(0,1)}[x_0, y_0] f^{(1,1)}[x_0, y_0] +$$
$$3 f^{(1,0)}[x_0, y_0] f^{(1,1)}[x_0, y_0] + 3 f[x_0, y_0]^2 f^{(1,2)}[x_0, y_0] +$$
$$f^{(0,1)}[x_0, y_0] f^{(2,0)}[x_0, y_0] + 3 f[x_0, y_0] f^{(2,1)}[x_0, y_0] + f^{(3,0)}[x_0, y_0]\Big)$$

In[5]:= $$k_1 = h f[x_0, y_0];$$
$$k_2 = h f\left[x_0 + \frac{h}{2},\; y_0 + \frac{1}{2} k_1\right];$$
$$k_3 = h f\left[x_0 + \frac{h}{2},\; y_0 + \frac{1}{2} k_2\right];$$
$$k_4 = h f[x_0 + h,\; y_0 + k_3];$$
$$y_1 = y_0 + \frac{1}{6}(k_1 + 2 k_2 + 2 k_3 + k_4)$$

Out[9]= $$\frac{1}{6} \Bigg(h f[x_0, y_0] + 2 h f\left[\frac{h}{2} + x_0,\; \frac{1}{2} h f[x_0, y_0] + y_0\right] +$$
$$2 h f\left[\frac{h}{2} + x_0,\; \frac{1}{2} h f\left[\frac{h}{2} + x_0,\; \frac{1}{2} h f[x_0, y_0] + y_0\right] + y_0\right] +$$
$$h f\left[h + x_0,\; h f\left[\frac{h}{2} + x_0,\; \frac{1}{2} h f\left[\frac{h}{2} + x_0,\; \frac{1}{2} h f[x_0, y_0] + y_0\right] + y_0\right] + y_0\right]\Bigg) + y_0$$

In[10]:= $$\text{ser2} = \text{Series}[y_1,\; \{h,\; 0,\; 4\}] \;//\; \text{Normal}$$

Out[10]= $$h f[x_0, y_0] + y_0 + \frac{1}{2} h^2 \left(f[x_0, y_0] f^{(0,1)}[x_0, y_0] + f^{(1,0)}[x_0, y_0]\right) +$$
$$\frac{1}{6} h^3 \Big(f[x_0, y_0] f^{(0,1)}[x_0, y_0]^2 + f[x_0, y_0]^2 f^{(0,2)}[x_0, y_0] +$$
$$f^{(0,1)}[x_0, y_0] f^{(1,0)}[x_0, y_0] + 2 f[x_0, y_0] f^{(1,1)}[x_0, y_0] + f^{(2,0)}[x_0, y_0]\Big) +$$
$$\frac{1}{24} h^4 \Big(f[x_0, y_0] f^{(0,1)}[x_0, y_0]^3 + 4 f[x_0, y_0]^2 f^{(0,1)}[x_0, y_0] f^{(0,2)}[x_0, y_0] +$$
$$f[x_0, y_0]^3 f^{(0,3)}[x_0, y_0] + f^{(0,1)}[x_0, y_0]^2 f^{(1,0)}[x_0, y_0] +$$
$$3 f[x_0, y_0] f^{(0,2)}[x_0, y_0] f^{(1,0)}[x_0, y_0] + 5 f[x_0, y_0] f^{(0,1)}[x_0, y_0] f^{(1,1)}[x_0, y_0] +$$
$$3 f^{(1,0)}[x_0, y_0] f^{(1,1)}[x_0, y_0] + 3 f[x_0, y_0]^2 f^{(1,2)}[x_0, y_0] +$$
$$f^{(0,1)}[x_0, y_0] f^{(2,0)}[x_0, y_0] + 3 f[x_0, y_0] f^{(2,1)}[x_0, y_0] + f^{(3,0)}[x_0, y_0]\Big)$$

In[11]:= $$\text{ser1} == \text{ser2}$$

Out[11]= True