Riddle Along. student: What's $(x^x)'$
Prof A. Like $(x^n)'$, $(x^x)' = x \cdot x^{x-1} = x^x$
Prof B. Like $(a^x)'$, $(x^x)' = x^x \log x$
Prof C. It's the sum of the two?
Prof D. Wow! There must be a reason?

$\phi' = f(x, \phi')$ f is unif. Lipschitz, Sol'n exist & is unique.

Eqn $\Leftrightarrow$ $\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\,dt$

Plan. $\phi_0(x) = y_0$ $\phi_n(x) = y_0 + \int_{x_0}^{x} f(t, \phi_{n-1}(t))\,dt$

Claims 1. ϕ_n is well defined.

2. For $n \geq 1$, $|\phi_n(x) - \phi_{n-1}(x)| < \frac{MK^{n-1}}{n!}|x - x_0|^n$

3. $\phi = \lim \phi_n$ converges uniformly.

start line

4. ϕ is a sol'n!

(3–4: leave for HW)

5. ϕ is unique; if ϕ, ψ both solve, then

$$|\phi(x) - \psi(x)| \leq A \cdot \int_0^x |\phi(t) - \psi(t)|\,dt$$

Set $U(x) = \int_0^x |\phi(t) - \psi(t)|\,dt$ then $U \geq 0$,

$U(0) = 0$ & $(e^{-Ax}U(x))' \leq 0$ so

$e^{-Ax}U(x) \leq 0$ so $U \equiv 0$. $\square$

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* Systems of equations.
* Higher order equations.

done line

An aside on $g(t)'$ where $g(t) = f(\gamma_1(t), \gamma_2(t), \gamma_3(t))$

When adding ϵ to t, g increases for 3 reasons; so

$$g'(t) = F_1 \cdot \gamma_1' + F_2 \cdot \gamma_2' + F_3 \cdot \gamma_3'$$

Example $(x^x)'$

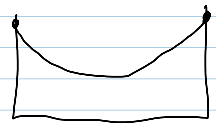
Calculus of Variations

= "infinite dimensional differential calculus"
= "solving min/max on function spaces"

<u>Problem</u> Minimize $J(y) = \int_a^b F(x, y, y')\,dx$ among all sufficiently differentiable $y(x)$ s.t.

$y(a) = A, \quad y(b) = B$

<u>Examples</u> 1. The power line problem

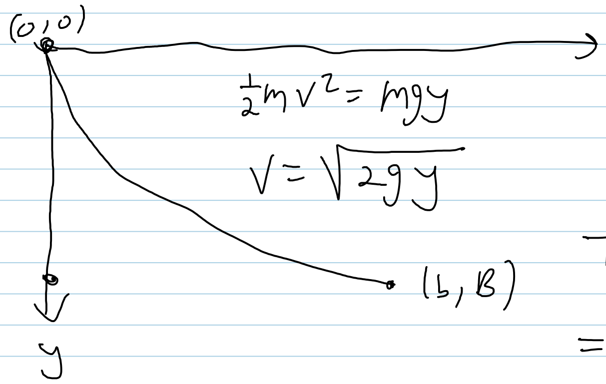


$$J(y) = \int y \cdot m\,ds = m\int y \cdot \sqrt{1+y'^2}\,dx$$

2. $\mathcal{L}(q) = \int \left[\frac{1}{2} m \dot{q}^2 - V(q(t))\right] dt$

(all of classical mechanics is hidden here)

3. The Brachistochrone.



$$\frac{1}{2} m v^2 = mgy$$

$$v = \sqrt{2gy}$$

$$ds = v\,dt = \sqrt{2gy}\,dt$$

$$ds = \sqrt{1+y'^2}\,dx$$

$$T(y) = \int dt = \int \frac{ds}{\sqrt{2gy}} = \int_0^b \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$$