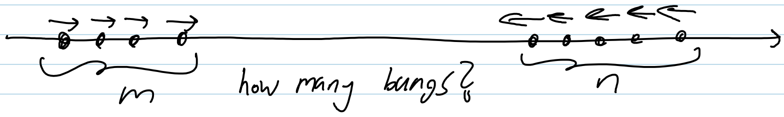
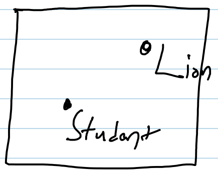
APUS

Putnam: Dec 1 2012, UofT.me/putnam

Read Along: BDP Section 2.6.

Riddle Along





$V_L = V_S$; Can S survive forever?

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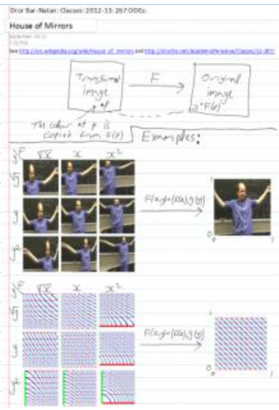
1. House of mirrors
2. Separated equations / gradient eqn's
3. More general gradient equations detecting "exactness"
4. Solving exact equations
5. Integration factors.



Go over handout. Suppose I wanted an eq'n whose solution is $\underbrace{f(x)+g(y)}_{}=C$,

what would I do? $\quad \Psi(x,y)=C$

$$(\nabla\Psi)\cdot\begin{pmatrix}1\\ y'\end{pmatrix}=0 \qquad \Psi_x+\Psi_y\,y'=0$$

If $\Psi=f(x)+g(y)$, this is "separated"

done line

Otherwise, it is "exact".

How can we tell if $M+Ny'=0$ or $M\,dx+N\,dy=0$ is exact?

<u>Claim</u> This is iff $M_y=N_x$.

More precisely, if on some rectangle M, N & their derivatives exist & are continuous then

$$\exists\,\Psi \text{ s.t. } \Psi_x=M,\ \Psi_y=N \iff M_y=N_x$$

<u>PF</u> $\Longrightarrow$ Easy

$\Leftarrow$ Suppose $\chi_x = M$. Wish to find $\phi(y)$ s.t. $\psi = \chi + \phi$

works: $(\chi + \phi)_y = N$ i.e. $\phi_y = N - \chi_y$;

possible if $0 = (N - \chi_y)_x = N_x - \chi_{xy} = N_x - M_y$

$M_y = \cos x + 2xe^y$ $N_x = \cos x + 2xe^y$!

$\chi = y\sin x + x^2 e^y$

$\phi = 2y$

דוגמא: פתור

$$\underbrace{(y\cos x + 2xe^y)}_{M} + \underbrace{(\sin x + x^2 e^y + 2)}_{N} y' = 0$$

(הפתרון: $\psi(x,y) = y\sin x + x^2 e^y + 2y$) אז נשים $x=0$, $y=1$ ...

6. גורמי אינטגרציה:

לפעמים $M + Ny' = 0$ הופכת מדויקת אם מכפילים ב-μ ...

התנאי על M:

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

זו מד״ח. יכולים להיות הרבה פתרונות, ויכול להיות מאוד קשה למצוא)

לעיתים קל אם יש הנחות:

μ depends only on y:

$$\frac{M\mu_y}{\mu} = N_x - M_y$$

$$\frac{\mu_y}{\mu} = \frac{N_x - M_y}{M}$$

depends only on y

μ תלוי רק ב-x:

$$\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}$$

ניתן לפתרון אם RHS לא ב-y.

דוגמא: פתור

$$\underbrace{(3xy + y^2)}_{M} + \underbrace{(x^2 + xy)}_{N} y' = 0$$

$$\frac{M_y - N_x}{N} = \frac{3x + 2y - 2x - y}{x^2 + xy} = \frac{x+y}{x(x+y)} = \frac{1}{x}$$

$$\frac{\mu_x}{\mu} = \frac{1}{x} \Rightarrow \mu = x$$

$$(3x^2 y + xy^2)\,dx + (x^3 + x^2 y)\,dy = 0$$

$$\psi = x^3 y + \tfrac{1}{2}x^2 y^2 + \phi(y)$$

$$\psi_y = x^3 + x^2 y + \phi' \qquad \psi = x^3 y + \tfrac{1}{2}x^2 y^2 = C$$