MAT 267 Advanced Ordinary Differential Equations

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This class will be videotaped, but you're making a huge mistake if you plan to skip all classes

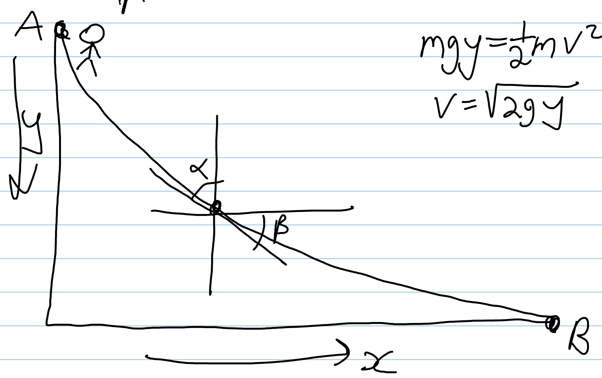
Differential Equation: The unknown is a function, what is given is a relation between the function and its derivatives:

1. $y' = y$ 2. $y' = y + e^x$ (sol'n $y = xe^x$)

3. $$\frac{y e^{(y-y')^2}}{\cos(x+y')} = y'' \quad (\text{sol'n}: ???)$$

We'll learn a bit about where these come from, how to solve them, and how to study their solutions.

Example. The Brachistochrone:



$mgy = \frac{1}{2} m v^2$

$v = \sqrt{2gy}$

later:

$\sin\alpha = \cos\beta =$

$$\frac{1}{\sqrt{1+\tan^2\beta}} = \frac{1}{\sqrt{1+y'^2}}$$

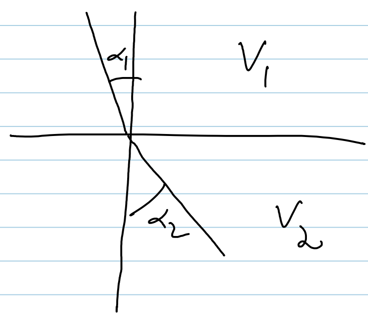
Fermat's principle: Light travels along the path of least time.

Snell's law:

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{V_1}{V_2}$$

or

$$\frac{V_1}{\sin\alpha_1} = \frac{V_2}{\sin\alpha_2}$$



$\Longrightarrow$ In our problem, $\frac{v}{\sin\alpha} = C$

So $\sqrt{2gy}\cdot\sqrt{1+y'^2} = C$ or better

done line

$y(0)=0$
$y(B)=A$

$y(1+y'^2) = d$ How solve? That's why we're here!

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In[1]:= DSolve[y[x] Sqrt[1 + (y'[x])^2] == C2, y[x], x]
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$$\text{Out[1]=}\ \Big\{\{y[x]\to -\sqrt{-x^2-2\,x\,C[1]-C[1]^2+C_2^2}\},\ \{y[x]\to \sqrt{-x^2-2\,x\,C[1]-C[1]^2+C_2^2}\},\ \{y[x]\to -\sqrt{-x^2+2\,x\,C[1]-C[1]^2+C_2^2}\},\ \{y[x]\to \sqrt{-x^2+2\,x\,C[1]-C[1]^2+C_2^2}\}\Big\}$$

Better, $1+y'^2 = d/y$

$$y' = \sqrt{\frac{d-y}{y}}$$

$$\frac{dy}{dx} = \sqrt{\frac{d-y}{y}}$$

$$\sqrt{\frac{y}{d-y}}\,dy = dx \Rightarrow \int\sqrt{\frac{y}{d-y}}\,dy = x + C_1$$

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In[5]:= ∫ Sqrt[y/(d - y)] dy // FullSimplify
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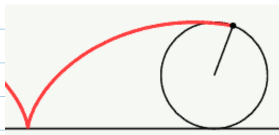
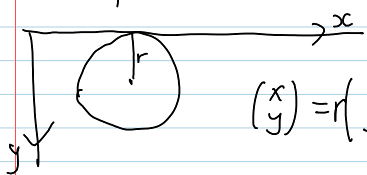
$$\text{Out[5]=}\ \frac{\sqrt{\frac{y}{-y+c_2}}\left(y^{3/2}+c_2\left(-\sqrt{y}+\text{ArcTan}\left[\frac{\sqrt{y}}{\sqrt{-y+c_2}}\right]\sqrt{-y+c_2}\right)\right)}{\sqrt{y}}$$

claim: This is a "Cycloid".





http://en.wikipedia.org/wiki/Cycloid

$$\binom{x}{y} = r\binom{-\sin t}{-\cos t} + \binom{tr}{r}$$

$$\rightsquigarrow r\binom{t-\sin t}{1-\cos t} \qquad x(t) = r(t-\sin t),\quad y(t) = r(1-\cos t)$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\left(\frac{dx}{dt}\right)^{-1} = \frac{\sin t}{1-\cos t}$$

Is there d for which $y' = \sqrt{\frac{d-y}{y}}$?

Works for $d=2r$:

$$\left(\frac{S}{1-C}\right)^2 \overset{?}{=} \frac{2r - r(1-C)}{r(1-C)} = \left(\frac{1+C}{1-C}\right) \qquad \frac{1-C^2}{(1-C)^2} \overset{?}{=} \frac{1+C}{1-C} \qquad \frac{\cancel{(1-C)}(1+C)}{(1-C)^{\cancel{2}}} = \frac{1+C}{1-C}\ \checkmark$$

Words: "Differential Equation"

"Ordinary"

"Partial"

"Order"

Linear $\to$ homogeneous

Linear $\to$ non-homogeneous