Penseive header: Finding the most general Runge-Kutta method.

```
n = 2; m = 2;

Clear[ϕ];
ϕ[x0] = y0;
ϕ0[x_] := ϕ[x];
ϕk_[x_] /; k ≥ 1 := ϕk[x] = Expand[
     ∂x (ϕk-1[x]) /. ϕ0'[x] :→ f[x, ϕ0[x]]
   ];

ExactSeries = Sum[1/k! ϕk[x] h^k, {k, 0, n}] /. x → x0 /. {f[x0, y0] → f0,0, f^(i_,j_)[x0, y0] :→ fi,j}
```

$$y_0 + h f_{0,0} + \frac{1}{2} h^2 (f_{0,0} f_{0,1} + f_{1,0})$$

```
k1 = f[x0, y0];
Table[
  kj = f[x0 + h γj, y0 + h Sum[αj,i ki, {i, 1, j-1}]],
  {j, 2, m}
 ];

y1 = y0 + h Sum[βj kj, {j, 1, m}];

ApproximationSeries =
 (Series[y1, {h, 0, n}] // Normal // Collect[#, h, Simplify] &) /.
  {f[x0, y0] → f0,0, f^(i_,j_)[x0, y0] :→ fi,j}
```

$$y_0 + h (\beta_1 + \beta_2) f_{0,0} + h^2 \beta_2 (\gamma_2 f_{1,0} + f_{0,0} f_{0,1} \alpha_{2,1})$$

```
Union[Cases[ExactSeries, f_,_, Infinity]]
```

$$\{f_{0,0}, f_{0,1}, f_{1,0}\}$$

```
sol = SolveAlways[
  ExactSeries == ApproximationSeries,
  Union[{h}, Cases[ExactSeries, f_,_, Infinity]]
 ]
```

$$\left\{\left\{\beta_1 \to 1 - \beta_2, \gamma_2 \to \frac{1}{2 \beta_2}, \alpha_{2,1} \to \frac{1}{2 \beta_2}\right\}\right\}$$

```
k_1 = -1;
Table[
  k_j = -h Sum[α_{j,i} k_i, {i, 1, j-1}],
  {j, 2, m}
];
```

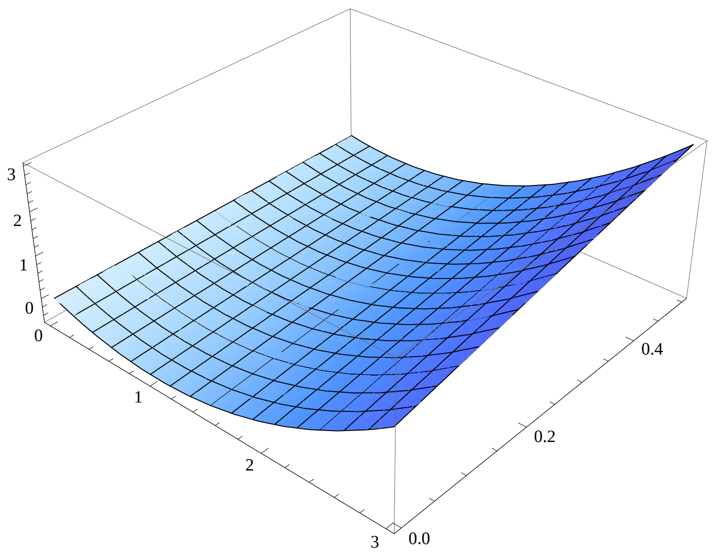
$$y_1 = h \sum_{j=1}^{m} \beta_j k_j$$

$h\,(-\beta_1 + h\,\beta_2\,\alpha_{2,1})$

`h (-β1 + h β2 α2,1) /. First[sol] /. β2 → β`

$h\left(-1 + \frac{h}{2} + \beta\right)$

`Plot3D[h (-1 + h/2 + β), {h, 0, 3}, {β, 0, 1/2}]`



`Solve[(h (-β1 + h β2 α2,1) /. First[sol]) == 1, h]`

$\left\{\left\{h \to 1 - \beta_2 - \sqrt{3 - 2\,\beta_2 + \beta_2^2}\right\}, \left\{h \to 1 - \beta_2 + \sqrt{3 - 2\,\beta_2 + \beta_2^2}\right\}\right\}$