Very basic Frobenius series manipulations

Some Definitions

```
FundamentalSeries[x^2 y'' + x P_. y' + Q_. y, n_] := FundamentalSeries[P, Q, n];
FundamentalSeries[x^2 y'' + Q_. y, n_] := FundamentalSeries[0, Q, n];
FundamentalSeries[x^2 y'' + x P_. y', n_] := FundamentalSeries[P, 0, n];
FundamentalSeries[P_, Q_, n_] := Module[{p, q, F, a},
  p_k_ := p_k = SeriesCoefficient[P, {x, 0, k}];
  q_k_ := q_k = SeriesCoefficient[Q, {x, 0, k}];
  F[α_] := α (α - 1) + p_0 α + q_0;
  Print[α /. Solve[F[α] == 0]];
  a_0 = 1;
  a_k_ /; k > 0 := a_k = (-Sum_{j=0}^{k-1} ((α + j) p_{k-j} + q_{k-j}) a_j) / F[α + k];
  ((Sum_{k=0}^{n} a_k x^k) + O[x]^(n+1)) x^α
 ];
L[eqn_, f_] := Simplify[eqn /. {y'' → D[f, x, x], y' → D[f, x], y → f}];
SimplifyCoefficients[expr_] := expr /. s_SeriesData :→ MapAt[Simplify, s, 3];
```

The Bessel Function J_0

```
eqn = x^2 y'' + x y' + x^2 y;
ϕ = FundamentalSeries[eqn, 10]
```

{0, 0}

$$x^\alpha\left(1-\frac{x^2}{(2+\alpha)^2}+\frac{x^4}{(2+\alpha)^2(4+\alpha)^2}-\frac{x^6}{(2+\alpha)^2(4+\alpha)^2(6+\alpha)^2}+\frac{x^8}{(2+\alpha)^2(4+\alpha)^2(6+\alpha)^2(8+\alpha)^2}-\frac{x^{10}}{(2+\alpha)^2(4+\alpha)^2(6+\alpha)^2(8+\alpha)^2(10+\alpha)^2}+O[x]^{11}\right)$$

```
L[eqn, ϕ]
```

$$x^\alpha\left(\alpha^2+O[x]^{11}\right)$$

```
y1 = ϕ /. α → 0
```

$$1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147\,456}-\frac{x^{10}}{14\,745\,600}+O[x]^{11}$$

```
L[eqn, y1]
```

$$O[x]^{11}$$

```
y2 = D[ϕ, α] /. α → 0
```

$$\text{Log}[x]+\left(\frac{1}{4}-\frac{\text{Log}[x]}{4}\right)x^2+\left(-\frac{3}{128}+\frac{\text{Log}[x]}{64}\right)x^4+\left(\frac{11}{13\,824}-\frac{\text{Log}[x]}{2304}\right)x^6+\left(-\frac{25}{1\,769\,472}+\frac{\text{Log}[x]}{147\,456}\right)x^8+\left(\frac{137}{884\,736\,000}-\frac{\text{Log}[x]}{14\,745\,600}\right)x^{10}+O[x]^{11}$$

```
L[eqn, y2]
```

$$O[x]^{11}$$

y2 - Log[x] y1

$$\frac{x^2}{4}-\frac{3x^4}{128}+\frac{11x^6}{13824}-\frac{25x^8}{1769472}+\frac{137x^{10}}{884736000}+O[x]^{11}$$

The Bessel Function $J_{1/3}$

eqn = x^2 y'' + x y' + (x^2 - 1/9) y;

ϕ = FundamentalSeries[eqn, 10]

$$\left\{-\frac{1}{3},\ \frac{1}{3}\right\}$$

$$x^{\alpha}\left(1-\frac{9x^2}{35+36\alpha+9\alpha^2}+\frac{81x^4}{(35+36\alpha+9\alpha^2)(143+72\alpha+9\alpha^2)}-\frac{729x^6}{(35+36\alpha+9\alpha^2)(143+72\alpha+9\alpha^2)(323+108\alpha+9\alpha^2)}+\left(6561x^8\right)/\left((35+36\alpha+9\alpha^2)(143+72\alpha+9\alpha^2)(323+108\alpha+9\alpha^2)(575+144\alpha+9\alpha^2)\right)-\left(59049x^{10}\right)/\left((35+36\alpha+9\alpha^2)(143+72\alpha+9\alpha^2)(323+108\alpha+9\alpha^2)(575+144\alpha+9\alpha^2)(899+180\alpha+9\alpha^2)\right)+O[x]^{11}\right)$$

L[eqn, ϕ]

$$x^{\alpha}\left(\left(-\frac{1}{9}+\alpha^2\right)+O[x]^{11}\right)$$

y1 = ϕ /. α → 1 / 3

$$x^{1/3}-\frac{3x^{7/3}}{16}+\frac{9x^{13/3}}{896}-\frac{9x^{19/3}}{35840}+\frac{27x^{25/3}}{7454720}-\frac{81x^{31/3}}{2385510400}+O[x]^{34/3}$$

y2 = ϕ /. α → -1 / 3

$$\frac{1}{x^{1/3}}-\frac{3x^{5/3}}{8}+\frac{9x^{11/3}}{320}-\frac{9x^{17/3}}{10240}+\frac{27x^{23/3}}{1802240}-\frac{81x^{29/3}}{504627200}+O[x]^{32/3}$$

y3 = y2 + $\frac{2x^{23/3}}{1802240}$

$$\frac{1}{x^{1/3}}-\frac{3x^{5/3}}{8}+\frac{9x^{11/3}}{320}-\frac{9x^{17/3}}{10240}+\frac{29x^{23/3}}{1802240}-\frac{81x^{29/3}}{504627200}+O[x]^{32/3}$$

Table[L[eqn, y_i], {i, 3}]

$$\left\{O[x]^{34/3},\ O[x]^{32/3},\ \frac{x^{23/3}}{15360}+\frac{x^{29/3}}{901120}+O[x]^{32/3}\right\}$$

The Bessel Function $J_{1/2}$

eqn = x^2 y'' + x y' + (x^2 - 1/4) y;

ϕ = FundamentalSeries[eqn, 10]

$$\left\{-\frac{1}{2},\ \frac{1}{2}\right\}$$

$$x^{\alpha}\left(1-\frac{4x^2}{15+16\alpha+4\alpha^2}+\frac{16x^4}{(15+16\alpha+4\alpha^2)(63+32\alpha+4\alpha^2)}-\frac{64x^6}{(15+16\alpha+4\alpha^2)(63+32\alpha+4\alpha^2)(143+48\alpha+4\alpha^2)}+\left(256x^8\right)/\left((15+16\alpha+4\alpha^2)(63+32\alpha+4\alpha^2)(143+48\alpha+4\alpha^2)(255+64\alpha+4\alpha^2)\right)-\left(1024x^{10}\right)/\left((15+16\alpha+4\alpha^2)(63+32\alpha+4\alpha^2)(143+48\alpha+4\alpha^2)(255+64\alpha+4\alpha^2)(399+80\alpha+4\alpha^2)\right)+O[x]^{11}\right)$$

```
L[eqn, ϕ]
```

$$x^{\alpha}\left(\left(-\frac{1}{4}+\alpha^2\right)+O[x]^{11}\right)$$

```
y₁ = ϕ /. α → 1 / 2
```

$$\sqrt{x}-\frac{x^{5/2}}{6}+\frac{x^{9/2}}{120}-\frac{x^{13/2}}{5040}+\frac{x^{17/2}}{362\,880}-\frac{x^{21/2}}{39\,916\,800}+O[x]^{23/2}$$

```
y₂ = ϕ /. α → -1 / 2
```

$$\frac{1}{\sqrt{x}}-\frac{x^{3/2}}{2}+\frac{x^{7/2}}{24}-\frac{x^{11/2}}{720}+\frac{x^{15/2}}{40\,320}-\frac{x^{19/2}}{3\,628\,800}+O[x]^{21/2}$$

```
{L[eqn, y₁], L[eqn, y₂]}
```

$$\left\{O[x]^{23/2},\ O[x]^{21/2}\right\}$$

The Bessel Function J_1

```
eqn = x^2 y'' + x y' + (x^2 - 1) y;
ϕ = FundamentalSeries[eqn, 10]
```

$\{-1, 1\}$

$$x^{\alpha}\left(1-\frac{x^2}{3+4\alpha+\alpha^2}+\frac{x^4}{(1+\alpha)(3+\alpha)^2(5+\alpha)}-\frac{x^6}{(1+\alpha)(3+\alpha)^2(5+\alpha)^2(7+\alpha)}+\frac{x^8}{(1+\alpha)(3+\alpha)^2(5+\alpha)^2(7+\alpha)^2(9+\alpha)}-\frac{x^{10}}{(1+\alpha)(3+\alpha)^2(5+\alpha)^2(7+\alpha)^2(9+\alpha)^2(11+\alpha)}+O[x]^{11}\right)$$

```
L[eqn, ϕ]
```

$$x^{\alpha}\left(\left(-1+\alpha^2\right)+O[x]^{11}\right)$$

```
y₁ = ϕ /. α → 1
```

$$x-\frac{x^3}{8}+\frac{x^5}{192}-\frac{x^7}{9216}+\frac{x^9}{737\,280}-\frac{x^{11}}{88\,473\,600}+O[x]^{12}$$

```
ϕ /. α → -1
```

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

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General::stop : Further output of Power::infy will be suppressed during this calculation. ≫

$$\frac{1}{x}+\text{ComplexInfinity}\,x+\text{ComplexInfinity}\,x^3+\text{ComplexInfinity}\,x^5+\text{ComplexInfinity}\,x^7+\text{ComplexInfinity}\,x^9+O[x]^{10}$$

```
(α + 1) ϕ
```

$$x^{\alpha}\left((1+\alpha)-\frac{(1+\alpha)x^2}{3+4\alpha+\alpha^2}+\frac{x^4}{(3+\alpha)^2(5+\alpha)}-\frac{x^6}{(3+\alpha)^2(5+\alpha)^2(7+\alpha)}+\frac{x^8}{(3+\alpha)^2(5+\alpha)^2(7+\alpha)^2(9+\alpha)}-\frac{x^{10}}{(3+\alpha)^2(5+\alpha)^2(7+\alpha)^2(9+\alpha)^2(11+\alpha)}+O[x]^{11}\right)$$

```
(α + 1) ϕ // SimplifyCoefficients
```

$$x^{\alpha}\left((1+\alpha)-\frac{x^2}{3+\alpha}+\frac{x^4}{(3+\alpha)^2\,(5+\alpha)}-\frac{x^6}{(3+\alpha)^2\,(5+\alpha)^2\,(7+\alpha)}+\frac{x^8}{(3+\alpha)^2\,(5+\alpha)^2\,(7+\alpha)^2\,(9+\alpha)}-\frac{x^{10}}{(3+\alpha)^2\,(5+\alpha)^2\,(7+\alpha)^2\,(9+\alpha)^2\,(11+\alpha)}+O[x]^{11}\right)$$

```
D[(α + 1) ϕ // SimplifyCoefficients, α]
```

$$x^{\alpha}\left(1+\frac{x^2}{(3+\alpha)^2}+\left(-\frac{1}{(3+\alpha)^2\,(5+\alpha)^2}-\frac{2}{(3+\alpha)^3\,(5+\alpha)}\right)x^4+\left(\frac{1}{(3+\alpha)^2\,(5+\alpha)^2\,(7+\alpha)^2}+\frac{2}{(3+\alpha)^2\,(5+\alpha)^3\,(7+\alpha)}+\frac{2}{(3+\alpha)^3\,(5+\alpha)^2\,(7+\alpha)}\right)x^6+\left(-\frac{1}{(3+\alpha)^2\,(5+\alpha)^2\,(7+\alpha)^2\,(9+\alpha)^2}-\frac{2}{(3+\alpha)^2\,(5+\alpha)^2\,(7+\alpha)^3\,(9+\alpha)}-\frac{2}{(3+\alpha)^2\,(5+\alpha)^3\,(7+\alpha)^2\,(9+\alpha)}-\frac{2}{(3+\alpha)^3\,(5+\alpha)^2\,(7+\alpha)^2\,(9+\alpha)}\right)x^8+\left(\frac{1}{(3+\alpha)^2\,(5+\alpha)^2\,(7+\alpha)^2\,(9+\alpha)^2\,(11+\alpha)^2}+\frac{2}{(3+\alpha)^2\,(5+\alpha)^2\,(7+\alpha)^2\,(9+\alpha)^3\,(11+\alpha)}+\frac{2}{(3+\alpha)^2\,(5+\alpha)^2\,(7+\alpha)^3\,(9+\alpha)^2\,(11+\alpha)}+\frac{2}{(3+\alpha)^2\,(5+\alpha)^3\,(7+\alpha)^2\,(9+\alpha)^2\,(11+\alpha)}+\frac{2}{(3+\alpha)^3\,(5+\alpha)^2\,(7+\alpha)^2\,(9+\alpha)^2\,(11+\alpha)}\right)x^{10}+O[x]^{11}\right)+$$

$$x^{\alpha}\left((1+\alpha)\,\mathrm{Log}[x]-\frac{\mathrm{Log}[x]\,x^2}{3+\alpha}+\frac{\mathrm{Log}[x]\,x^4}{(3+\alpha)^2\,(5+\alpha)}-\frac{\mathrm{Log}[x]\,x^6}{(3+\alpha)^2\,(5+\alpha)^2\,(7+\alpha)}+\frac{\mathrm{Log}[x]\,x^8}{(3+\alpha)^2\,(5+\alpha)^2\,(7+\alpha)^2\,(9+\alpha)}-\frac{\mathrm{Log}[x]\,x^{10}}{(3+\alpha)^2\,(5+\alpha)^2\,(7+\alpha)^2\,(9+\alpha)^2\,(11+\alpha)}+O[x]^{11}\right)$$

```
y2 = D[(α + 1) ϕ // SimplifyCoefficients, α] /. α → -1
```

$$\frac{1}{x}+\left(\frac{1}{4}-\frac{\mathrm{Log}[x]}{2}\right)x+\left(-\frac{5}{64}+\frac{\mathrm{Log}[x]}{16}\right)x^3+\left(\frac{5}{1152}-\frac{\mathrm{Log}[x]}{384}\right)x^5+\left(-\frac{47}{442\,368}+\frac{\mathrm{Log}[x]}{18\,432}\right)x^7+\left(\frac{131}{88\,473\,600}-\frac{\mathrm{Log}[x]}{1\,474\,560}\right)x^9+O[x]^{10}$$

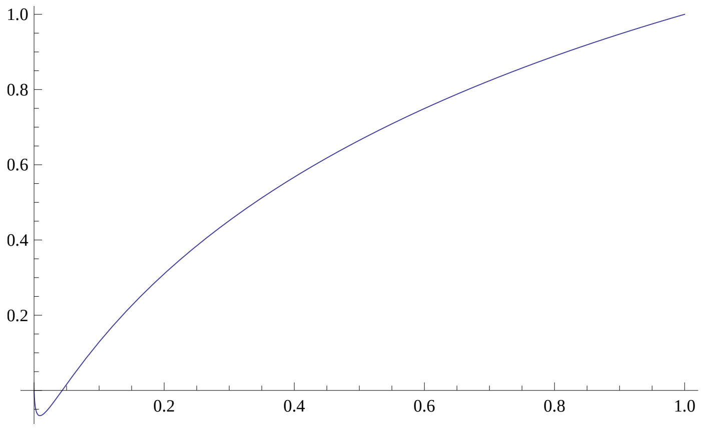
```
{L[eqn, y1], L[eqn, y2]}
```

$$\{O[x]^{12},\ O[x]^{10}\}$$

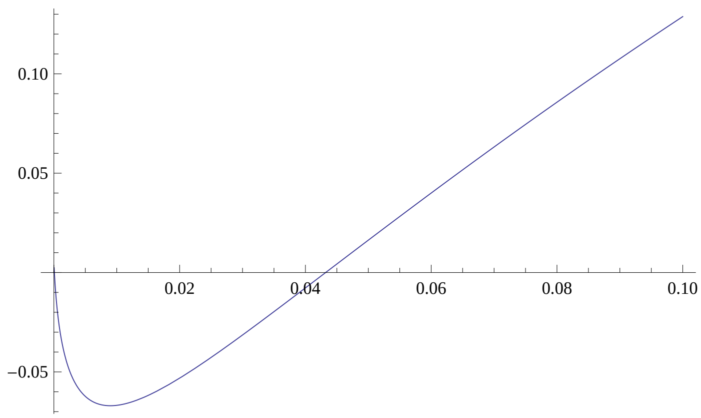
```
y2 + 1/2 Log[x] y1
```

$$\frac{1}{x}+\frac{x}{4}-\frac{5\,x^3}{64}+\frac{5\,x^5}{1152}-\frac{47\,x^7}{442\,368}+\frac{131\,x^9}{88\,473\,600}+O[x]^{10}$$

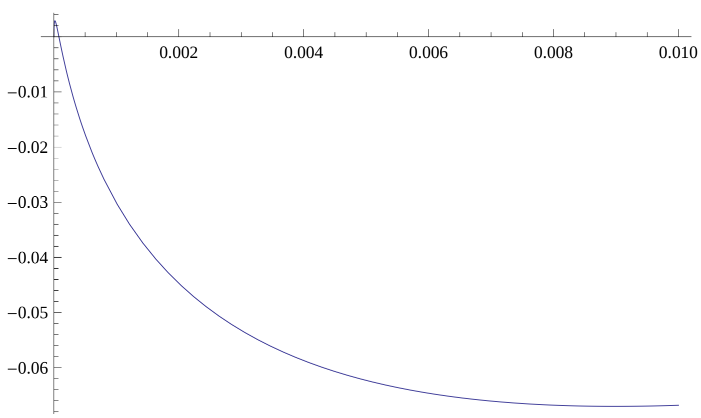
```
Plot[√x Cos[1/2 Log[x]], {x, 10^-12, 1}]
```




```
Plot[√x Cos[1/2 Log[x]], {x, 10^-12, 0.1}]
```




```
Plot[√x Cos[1/2 Log[x]], {x, 10^-12, 0.01}]
```



```
Plot[√x Cos[1/2 Log[x]], {x, 10^-12, 0.0001}]
```