Do not turn this page until instructed.

Math 267 Advanced Ordinary Differential Equations

Term Test

University of Toronto, October 26, 2012

Solve the 3 of the 4 problems on the other side of this page.
Each problem is worth 34 points.
You have fifty minutes to write this test.

$$\frac{\delta I}{\delta \varphi_i(x)} \equiv \frac{\partial \mathcal{L}(x)}{\partial \varphi_i(x)} - \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial [\partial_\mu \varphi_i(x)]} = 0$$

The field theoretic Euler-Lagrange equations, following Itzykson and Zuber.
It actually makes sense.

Notes.

- No outside material other than stationary and a basic calculator (not capable of displaying text) is allowed.
- If you solve all 4 problems, you must indicate clearly which 3 are to be graded. Failing this an arbitrary problem will be ignored in grading.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and made of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs and calculations given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.

Good Luck!

Solve 3 of the following 4 problems. Each problem is worth 34 points. You have fifty minutes. **Neatness counts! Language counts!**

Problem 1. Solve the following two differential equations:

1. $y'' - 2y' + 5y = 2x$ (find all solutions).
2. $y' = -\dfrac{x}{y}$ with $y(0) = 1$.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Problem 2.

1. State precisely (without proof) the theorem about existence and uniqueness of solutions for a single first order ordinary differential equation.
2. Show by an example that if the Lipschitz condition is dropped from the above statement, the uniqueness of solutions may fail.

Problem 3. Extremize the functional $J(Y) = \displaystyle\int_{-1}^{1} \frac{(y')^2}{x^3} dx$, subject to the boundary conditions $y(\pm 1) = 2$.

Problem 4.

1. State what is the "improved Euler method" for solving the differential equation $\phi' = f(x, \phi)$ with initial condition $\phi(x_0) = y_0$ using step size h.
2. Compute the single-step approximation for $y(1/3)$, given that y satisfies $y' = -\dfrac{x}{y}$ and $y(0) = 1$, using the improved Euler method.

Good Luck!