Math 267 Advanced Ordinary Differential Equations

Sample Final Exam

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Solve all of the following 5 questions. The questions carry equal weight though different parts of the same question may be weighted differently.

Duration. You have 3 hours to write this exam.

Allowed Material. Basic calculators, not capable of displaying text or sounding speech.

Good Luck!

Problem 1. Find the most general solutions of the following differential equations:

1. $y(y+1)dx + x(x-1)dy = 0$.
2. (pick your further favourites from HW or BDP).
3. ...
4. ...

Tip. All explicit integrations that are required above (and elsewhere in this exam) are easy; do not leave them un-evaluated.

Tip. It is always an excellent idea to substitute your solutions back into the equations and see if they really work.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Problem 2. A function $f(x, y)$ is continuous in some open set that contains a given point (x_0, y_0), and in the vicinity of x_0 a sequence of functions $\phi_n(x)$ is given by the recursive definition

$$\phi_0(x) = y_0, \qquad \phi_n(x) = y_0 + \int_{x_0}^{x} f(t, \phi_{n-1}(t))dt.$$

You may assume that somebody else had already proven that $\phi_n(x)$ is well defined.

Define what it means for the function f to be uniformly Lipschitz in the variable y, and prove that the sequence $\phi_n(x)$ is uniformly Cauchy and that it converges to a solution of the differential equation $y' = f(x, y)$ which passes through the point (x_0, y_0).

Tip. Neatness, cleanliness and organization count, here and everywhere else!

Problem 3. How would you solve a functional minimization problem for a functional of the form $J(y) = \int_a^b F(x, y, y', y'')dx$ (assuming $y(a) = A$ and $y(b) = B$)? Note that F involves also the *second* derivative y'' of y!

Problem 4. Find two Frobenius-series solutions of the Bessel equation of order 0, $x^2y'' + xy' + x^2y = 0$ near $x = 0$. There is no need to write a formula for the general terms in the solutions, but it is necessary to display enough terms to make the pattern clear.

Problem 5.

1. State and prove a theorem about oscillatory behaviour in the case when $\int_A^\infty q(x)dx = \infty$.
2. Use it to decide whether solutions of the Airy equation $y'' + xy = 0$ oscillate as $x \to \infty$.

Good Luck!