

Pensieve header: Ultra-fine oscillation criteria.

$$\text{eq} = \mathbf{y}' '[\mathbf{x}] + \gamma \mathbf{y}[\mathbf{x}] /. \mathbf{y} \rightarrow (\mathbf{Y}[\mathbf{e}^\#] \&) /. \\ \mathbf{x} \rightarrow \text{Log}[\mathbf{z}];$$

$$\text{eq} = \text{Expand}\left[\frac{\text{eq}}{\text{Coefficient}[\text{eq}, \mathbf{Y}' '[\mathbf{z}]]}\right]$$

$$\frac{\gamma \mathbf{Y}[\mathbf{z}]}{z^2} + \frac{\mathbf{Y}' [\mathbf{z}]}{z} + \mathbf{Y}'' [\mathbf{z}]$$

$$\{\mathbf{p}, \mathbf{q}\} = \text{Coefficient}[\text{eq}, \#] \& /@ \\ \{\mathbf{Y}' [\mathbf{z}], \mathbf{Y}[\mathbf{z}]\};$$

$$\mathbf{Q} = \mathbf{q} - \frac{1}{4} \mathbf{p}^2 - \frac{1}{2} \partial_z \mathbf{p} // \text{Simplify}$$

$$\frac{1 + 4 \gamma}{4 z^2}$$

$$\text{Eq0} = \mathbf{x}^2 \mathbf{y}' '[\mathbf{x}] + \gamma \mathbf{y}[\mathbf{x}]$$

$$\gamma \mathbf{y}[\mathbf{x}] + \mathbf{x}^2 \mathbf{y}'' [\mathbf{x}]$$

$$\text{Eq1} = \text{Eq0} /. \mathbf{y} \rightarrow (\mathbf{Y}[\mathbf{E}^\#] \&) /. \mathbf{x} \rightarrow \text{Log}[\mathbf{z}]$$

$$\gamma \mathbf{Y}[\mathbf{z}] + \text{Log}[\mathbf{z}]^2 (z \mathbf{Y}' [\mathbf{z}] + z^2 \mathbf{Y}'' [\mathbf{z}])$$

$$\text{Eq2} = \text{Expand}\left[\frac{\text{Eq1}}{\text{Coefficient}[\text{Eq1}, \mathbf{Y}' '[\mathbf{z}]]}\right]$$

$$\frac{\gamma \mathbf{Y}[\mathbf{z}]}{z^2 \text{Log}[\mathbf{z}]^2} + \frac{\mathbf{Y}' [\mathbf{z}]}{z} + \mathbf{Y}'' [\mathbf{z}]$$

$$\{\mathbf{p}, \mathbf{q}\} = \text{Coefficient}[\text{Eq2}, \#] \& /@ \{\mathbf{Y}' [\mathbf{z}], \mathbf{Y}[\mathbf{z}]\}$$

$$\left\{\frac{1}{z}, \frac{\gamma}{z^2 \text{Log}[\mathbf{z}]^2}\right\}$$

$$\mathbf{Q} = \mathbf{q} - \frac{1}{4} \mathbf{p}^2 - \frac{1}{2} \partial_z \mathbf{p}$$

$$\frac{1}{4 z^2} + \frac{\gamma}{z^2 \text{Log}[\mathbf{z}]^2}$$

$$\text{Eq3} = \mathbf{v}' '[\mathbf{z}] + \mathbf{Q} \mathbf{v}[\mathbf{z}]$$

$$\left(\frac{1}{4 z^2} + \frac{\gamma}{z^2 \text{Log}[\mathbf{z}]^2}\right) \mathbf{v}[\mathbf{z}] + \mathbf{v}'' [\mathbf{z}]$$

$$\text{Eq3} /. \{\mathbf{v} \rightarrow \mathbf{y}, \mathbf{z} \rightarrow \mathbf{x}\}$$

$$\left(\frac{1}{4 x^2} + \frac{\gamma}{x^2 \text{Log}[\mathbf{x}]^2}\right) \mathbf{y}[\mathbf{x}] + \mathbf{y}'' [\mathbf{x}]$$

```

UltraFine[0] = x^2 y''[x] + γ y[x];
UltraFine[n_] /; n > 0 := (
  Eq0 = UltraFine[n - 1];
  Eq1 = Eq0 /. y -> (Y[E^#] &) /. x -> Log[z];
  Eq2 = Expand[ $\frac{\text{Eq1}}{\text{Coefficient}[\text{Eq1}, \text{Y}'[z]]}$ ];
  {p, q} = Coefficient[Eq2, #] & /@ {Y'[z], Y[z]};
  Q = q -  $\frac{1}{4} p^2 - \frac{1}{2} \partial_z p$ ;
  Eq3 = v''[z] + Q v[z];
  Eq3 /. {v -> y, z -> x}
);

```

UltraFine[1]

$$\left(\frac{1}{4 x^2} + \frac{\gamma}{x^2 \text{Log}[x]^2} \right) y[x] + y''[x]$$

UltraFine[2]

$$\left(\frac{1}{4 x^2} + \frac{1}{4 x^2 \text{Log}[x]^2} + \frac{\gamma}{x^2 \text{Log}[x]^2 \text{Log}[\text{Log}[x]]^2} \right) y[x] + y''[x]$$

UltraFine[3]

$$\left(\frac{1}{4 x^2} + \frac{1}{4 x^2 \text{Log}[x]^2} + \frac{1}{4 x^2 \text{Log}[x]^2 \text{Log}[\text{Log}[x]]^2} + \frac{\gamma}{x^2 \text{Log}[x]^2 \text{Log}[\text{Log}[x]]^2 \text{Log}[\text{Log}[\text{Log}[x]]]^2} \right) y[x] + y''[x]$$