Fuchs' Theorem

Following Taylor's *Introduction to Differential Equations.*

Lazarus Immanuel Fuchs, 1833–1902

Theorem 1. *Suppose the series $v(x)=\sum_{k=0}^\infty v_kx^k$ solves the n-dimensional system $v'(x)=A(x)v(x)+g(x)$, where $A(x)$ and $g(x)$ are given by power series $A(x)=\sum_{k=0}^\infty A_kx^k$ and $g(x)=\sum_{k=0}^\infty g_kx^k$ that converge at radius R for some $R>0$. Then the series $v(x)$ converges for any x with $|x|<R$.*

Proof. Below $||M||$ where M is a matrix or a vector means "the largest absolute value of an entry of M".

The convergence of the series for A and for g implies that there are constants α and γ such that

$$||A_k||<\alpha R^{-k}\qquad\text{and}\qquad ||g_k||<\gamma R^{-k}.$$

We wish to show that whenever $r<R$, there is a constant η such that

$$||v_j||<\eta r^{-j}. \tag{1}$$

This we shall do by the method of "induction with an undetermined hypothesis". Namely, we assume that for some k Equation (1) holds for all $j\leq k$, without specifying η. We then prove that (1) is true for $j=k+1$ and see what conditions this may put on η. We keep track of these conditions, and at the end of the proof we verify that we could have satisfied them at the start of the proof.

The equation $v'=g+Av$ implies that $(k+1)v_{k+1}=g_k+\sum_{j=0}^k A_{k-j}v_j$. Therefore

$$(k+1)||v_{k+1}||\leq ||g_k||+\sum_{j=0}^k||A_{k-j}v_j||\leq ||g_k||+n\sum_{j=0}^k||A_{k-j}||\cdot||v_j||$$

$$<\gamma R^{-k}+n\sum_{j=0}^k\alpha R^{j-k}\cdot\eta r^{-j}=\gamma R^{-k}+n\alpha\eta r^{-k}\sum_{j=0}^k\left(\frac{r}{R}\right)^{k-j}.$$

The last sum is a geometric sum with ratio smaller than 1. Hence its value is bounded by some fixed constant β. Hence

$$(k+1)||v_{k+1}||<\gamma R^{-k}+\alpha\eta n\beta r^{-k}<r^{-k}(\gamma+\alpha\eta n\beta),$$

and thus, assuming $\eta\geq\gamma$,

$$||v_{k+1}||<r^{-(k+1)}\frac{r(\gamma+\alpha\eta n\beta)}{k+1}\leq\eta r^{-(k+1)}\frac{r(1+\alpha n\beta)}{k+1}.$$

Now for large enough k, say for $k>K$, the ugly fraction in the last formula will be smaller than 1, and we will have proven Equation (1) for $j=k+1$. We still need to make sure that Equation 1 holds for $j\leq K$. But this places only finitely many conditions on η, so we just need to pick η so that

$$\eta>\max\left(\gamma,r^j||v_j||\right)_{j\leq K}.$$

□

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