UNIVERSITY OF TORONTO

Faculty of Arts and Sciences

DECEMBER EXAMINATIONS 2012

Math 267H1 Advanced Ordinary Differential Equations — Final Exam

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Solve all of the following 5 questions. The questions carry equal weight though different parts of the same question may be weighted differently.

Duration. You have 3 hours to write this exam.

Allowed Material. Basic calculators, not capable of displaying text or sounding speech.

Good Luck!

Post-exam modifications in blue.

Solve all of the following 5 problems. Each problem is worth 20 points. You have three hours. **Neatness counts! Language counts!**

Problem 1. Find the most general solutions of the following differential equations:

1. $\dfrac{dy}{dt} + \dfrac{y}{t^2} = \dfrac{1}{t^2}$. (I should have asked this with $1/t^3$ on the right hand side).
2. $\dfrac{dy}{dx} - \dfrac{y}{x} = \dfrac{y^2}{x}$.
3. $(\sin y)dx + [(x+y)\cos y + \sin y]dy = 0$.
4. $y^{(4)} - 2y'' + y = 0$.

Tip. All explicit integrations that are required above (and elsewhere in this exam) are easy; do not leave them un-evaluated.

Tip. It is always an excellent idea to substitute your solutions back into the equations and see if they really work.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Problem 2.

1. State precisely (without proof) the theorem about existence and uniqueness of solutions for a single first order ordinary differential equation.
2. Show by an example that if the Lipschitz condition is dropped from the above statement, the uniqueness of solutions may fail.

Problem 3. It is given that a differentiable function $\phi\colon [a,b] \to \mathbb{R}$ is a local minimum of the functional $J(y) = \int_a^b F(x, y, y')dx$ among all differentiable functions y that satisfy $y(a) = A$ and $y(b) = B$, in the sense that whenever a differentiable function $h\colon [a,b] \to \mathbb{R}$ satisfies $h(a) = 0 = h(b)$, the function $\epsilon \to J(\phi + \epsilon h)$ has a local minimum at $\epsilon = 0$. Derive the Euler-Lagrange necessary condition that ϕ must satisfy.

~~**Tip.** Neatness, cleanliness and organization count, here and everywhere else!~~

Problem 4.

1. In this part t is always assumed to belong to some fixed interval I in $\mathbb{R}$. Assume $v_1(t)$ and $v_2(t)$ are differentiable $\mathbb{R}^2$-valued functions, are linearly independent for all t in I, and solve the system of differential equations $v'(t) = A(t)v(t)$, where $A(t)$ is a 2×2 matrix that depends continuously on t. Explain how by the method of "fundamental matrices", v_1 and v_2 can be used to solve the non-homogeneous version $v'(t) = A(t)v(t) + g(t)$ of the equation. (Here $g(t)$ is a given continuous $\mathbb{R}^2$-valued function).

2. Assume $t > 0$. For the following equation,

$$v' = \frac{1}{t}\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} v + \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

it is given that a solution of the homogeneous version is

$$v(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^{-1}.$$

Use the technique from the previous part of this question to find a solution of the full equation.

Tip. You may find it handy to note that $\left(\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right)^{-1} = \frac{1}{2}\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}.$

Problem 5.

1. State and prove the "Sturm Comparison Theorem".

2. Use it to decide whether solutions of $y'' + x^{-3/2}y = 0$ oscillate as $x \to \infty$.

Good Luck!