1

Frobenius series

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$$\sum_{k=0}^{n} y^{(n)}(x)\, p_k(x) = 0, \qquad p_n(x) = 1.$$

If p_k are analytic in nbds of 0, then 0 is an ordinary point. If not ordinary but $p_k x^{n-k}$ are, then 0 is a regular singular point. Otherwise, irregular singular point

e.g. $x^2 y'' + xy' = y$ regular singular $x=0$
$x^3 y' = (x+1)y$ irregular singular $x=0$

Fuchs: regular singular point then $\exists$ solution of the form $y = x^\alpha A(x)$, A analytic (and take $A(0)\neq 0$)

e.g. $y'' + \frac{y}{4x^2} = 0$. Using power series we get

$(n-\frac{1}{2})^2 a_n = 0 \qquad n = 0, 1, 2, \ldots$, hence $y(x) = 0$. But we want to find two linearly independent solutions.

We know by Fuchs that an ODE with a regular singular point at 0 has a Frobenius series solution,

$$y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n$$

For $y'' + \frac{y}{4x^2} = 0$, we get $\left[(n+\alpha)(n+\alpha-1) + \frac{1}{4}\right] a_n = 0$,

$n = 0, 1, \ldots$ $a_0 \neq 0$, so $\alpha(\alpha-1) + \frac{1}{4} = 0$, so $\alpha = \frac{1}{2}$

$(\alpha^2 - \alpha + \frac{1}{4} = 0$, $\alpha = \frac{1 \pm \sqrt{1-1}}{2} = \frac{1}{2})$

With α fixed, this forces $a_n = 0$, $n \geq 1$. So $y(x) = a_0\sqrt{x}$.

Say we have the eqn $y'' + \frac{p(x)}{x} y' + \frac{q(x)}{x^2} y = 0$
and it has a regular singular point at $x=0$. Then
p, q are analytic at $x=0$. So

$$p(x) = \sum_{n=0}^{\infty} p_n x^n, \quad q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

If we take y as a Frobenius series $y = x^{\alpha} \sum_{n=0}^{\infty} a_n x^n$,
then

$$y' = \sum_{n=0}^{\infty} a_n (\alpha+n) x^{\alpha+n-1} \quad y'' = \sum_{n=0}^{\infty} a_n (\alpha+n)(\alpha+n-1) x^{\alpha+n-2}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n (\alpha+n)(\alpha+n-1) x^{\alpha+n} + \sum_{n=0}^{\infty} p_n x^n \sum_{n=0}^{\infty} a_n (\alpha+n) x^{\alpha+n}$$
$$+ \sum_{n=0}^{\infty} q_n x^n \sum_{n=0}^{\infty} a_n x^{n+\alpha} = 0$$

x^{α} : $a_0 \alpha(\alpha-1) + p_0 a_0 \alpha + q_0 a_0 = 0$

$\Rightarrow \alpha^2 - \alpha + p_0 \alpha + q_0 = 0$

Indicial polynomial $P(\alpha) = \alpha^2 + (p_0 - 1)\alpha + q_0$

α is a root of $P(\alpha)$.

$x^{\alpha+n}$: $a_n (\alpha+n)(\alpha+n-1) + \sum_{k=0}^{n} p_{n-k} a_k (\alpha+k)$
$+ \sum_{k=0}^{n} q_{n-k} a_k = 0$

$$\Rightarrow a_n(\alpha+n)(\alpha+n-1) + p_0 a_n (\alpha+n) + q_0 a_n$$
$$= -\sum_{k=0}^{n-1} a_k [(\alpha+k) p_{n-k} + q_{n-k}]$$

$$\Rightarrow P(\alpha+n)a_n = -\sum_{k=0}^{n-1} a_k[(\alpha+k)p_{n-k} + q_{n-k}].$$

If $P(\alpha+n) \neq 0$ for $n=1,2,\ldots$ then we can solve for a_n, $n=1,2,\ldots$ to determine $y(x) = x^\alpha \sum_{n=0} a_n x^n$.

(Radius of conv. of A determined thusly is $\geq$ than dist. from 0 to nearest singularity of $p(x)$ or $q(x)$.)

Let α_1, α_2 be roots of $P(\alpha)$, $\mathrm{Re}\,\alpha_1 \geq \mathrm{Re}\,\alpha_2$. Then $P(\alpha_1+n) \neq 0$ $n=1,2,\ldots$ Then certainly we can determine $A(x)$, showing that there is always one Frobenius series solution.

e.g. $y'' + \frac{1}{x}y' - \left(1 + \frac{\nu^2}{x^2}\right)y = 0$ (modified Bessel eqn of order ν)

didn't do

$x=0$ regular singular point.

$P(\alpha) = \alpha^2 - \nu^2$, $\alpha = \pm\nu$, say $\nu \geq 0$, and $\alpha_1 = \nu$, $\alpha_2 = -\nu$.

We get $a_1 = a_3 = a_5 = \cdots = 0$, and

$$a_{2n} = \frac{a_{2n-2}}{2^2 n(\nu+n)} = \cdots = \frac{a_0\,\Gamma(\nu+1)}{2^{2n} n!\,\Gamma(\nu+n+1)}$$

Thus $$y(x) = a_0\,\Gamma(\nu+1)\,x^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}x\right)^{2n}}{n!\,\Gamma(\nu+n+1)}$$

As long as $\nu \neq 0,1,2,\ldots$, then doing the same game for $\alpha_2 = -\nu$ will give a Frobenius series that starts with $x^{-\nu}$ (if α_2 is a negative integer then $\Gamma(1+\alpha_2)$ will be infinite).

To find a second solution there are ~~three~~ four possibilities.

- If $\alpha_1 \neq \alpha_2 + n$ for any n, then we can play the same game again to get an independent solution which is a Frobenius series
- If $\alpha_1 = \alpha_2 + n$ for some particular $n > 0$ but $\sum_{k=0}^{n-1} a_k[(\alpha+k)p_{n-k} + q_{n-k}) = 0$. Then a_n is free, and since P has two roots it will never again be 0. This determines all the following a_k, $k = n+1, n+2, \ldots$, in terms of a_0 and a_n. Then again there is an independent Frobenius series solution.

$$\begin{cases} \bullet\ \alpha_1 = \alpha_2 \\ \bullet\ \alpha_1 - \alpha_2 = n \text{ for some } n > 0 \text{ and } \sum_{k=0}^{n-1} a_k[(\alpha+k)p_{n-k} \\ + q_{n-k}] \neq 0 \text{ for this } n \end{cases}$$

In these cases the second solution cannot be found in the form of a Frobenius series.

$\alpha_1 = \alpha_2$

If $y(x) = x^{\alpha_1} \sum_{n=0}^{\infty} a_n x^n$ solves $y'' + \frac{p(x)}{x} y' + \frac{q(x)}{x^2} y = 0$

then, if $y(x,\alpha) = x^{\alpha} \sum_{n=0}^{\infty} a_n(\alpha) x^n$, where $a_n(\alpha_1) = a_n$,
defined by recurrence

and let $L = \frac{d^2}{dx^2} + \frac{p(x)}{x}\frac{d}{dx} + \frac{q}{x^2}$, we get

like diff. of formal polynomial

$$(Ly)(x,\alpha) = a_0 x^{\alpha-2} P(\alpha)$$

silly trick $\frac{d}{d\alpha}$ and evaluate at $\alpha = \alpha_1$.

RHS $=0$ because double root $\alpha = \alpha_1 = \alpha_2$.

LHS: $\frac{d}{d\alpha}\left[(Ly)(x,\alpha) \right]\Big|_{\alpha=\alpha_2}$

$= L\left[\left(\frac{d}{d\alpha} y\right)\right](x,\alpha_2)$

Thus $\left(\frac{d}{d\alpha} y\right)(x,\alpha_2)$ is a solution to the ODE, as L sends it to 0.

Now, $= x^\alpha \sum_{n=0}^{\infty} a_n(\alpha) x^n$, so $\left(\frac{d}{d\alpha} y\right)(x,\alpha)$

$= x^\alpha (\log x) \sum_{n=0}^{\infty} a_n(\alpha) x^n + x^\alpha \sum_{n=0}^{\infty} a_n'(\alpha) x^n$

Let $b_n = \left(\frac{d}{d\alpha} a_n\right)(\alpha_2)$

Then $\left(\frac{dy}{d\alpha}\right)(x,\alpha_2) = \log(x)\, y(x,\alpha_2) + \sum_{n=0}^{\infty} b_n x^{n+\alpha_2}$

The new solution y_2 has radius of conv. at least as large as distance to nearest singularity of the ODE.

didn't do the following example

e.g. modified Bessel function of order 0:

$y'' + \frac{1}{x} y' - y = 0$. $P(\alpha) = \alpha^2 - \nu^2$, so double root $\alpha = 0$. We found one Frobenius series solution. The other won't be a Frobenius series but will have form

$$y_2(x) = \log(x)\, y_1(x) + \sum_{n=0}^{\infty} b_n x^n$$

$$b_n = \frac{d}{d\alpha} a_n(\alpha) \Big|_{\alpha=0}$$

$$\cancel{x^{\alpha+n-2} : [(\alpha+n)^2 - n^2] a_n(\alpha) = a_{n-2}(\alpha) \quad n=2,3,\ldots}$$

$(Ly)(x,\alpha) = a_0 \, x^{\alpha} p(\alpha)$ $\quad q(x) = -x^2$, so $q_0 = 0, q_1 = 0, q_2 = -1$

$\Rightarrow x^{\alpha-2}$: $a_0(\alpha) = a_0$

$x^{n+\alpha-2}$: $\quad p_0 = 1, q_0 = -1$ so $[(\alpha+n)^2 - 1] a_n(\alpha)$

$= -q_2 a_{n-2}(\alpha) \quad n \geq 2.$

$a_1(\alpha) = 0 \Rightarrow$ odd $a_n(\alpha) = 0$

And for $n = 1, 2, \ldots,$

$$a_{2n}(\alpha) = \frac{a_0}{(\alpha+2n)^2 \cdots (\alpha+2)^2}$$

Determine using logarithmic derivative.

$$b_{2n} = \frac{-a_0}{2^{2n}(n!)^2} H_n, \; n \geq 1, \; H_n = \sum_{k=1}^{n} \frac{1}{k}$$

Hence $y_2(x) = a_0 \log x \sum_{n=0}^{\infty} \frac{(\frac{1}{2}x)^{2n}}{(n!)^2} - a_0 \sum_{n=1}^{\infty} \frac{(\frac{1}{2}x)^{2n}}{(n!)^2} H_n$

Certainly not a constant multiple of $y_1(x)$, so linearly independent.

For $a_0 = -1$, if we add a certain mult. of y_1 we get $K_0(x)$ (another standard Bessel Function)