240Algebral-121204, Hours 35-36: Diagonalization

December-02-12
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Discussion of the Final. HW9 is due.

Do the course evals!

Problem. For any $A \in M_{n \times n}(F)$, compute A^p.

Example: $\begin{pmatrix} 4 & 3 \\ -6 & -5 \end{pmatrix}^{15} = ?$ [here $C^{-1}AC = D$, w/ $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$]

Brilliant idea: If $A = CDC^{-1}$ for invertible C & a diagonal D, $A^{15} = CD^{15}C^{-1}$

But how do we find C & D?

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad C = \begin{pmatrix} v_1 \,|\, v_2 \end{pmatrix} \qquad AC = CD \quad (*)$$

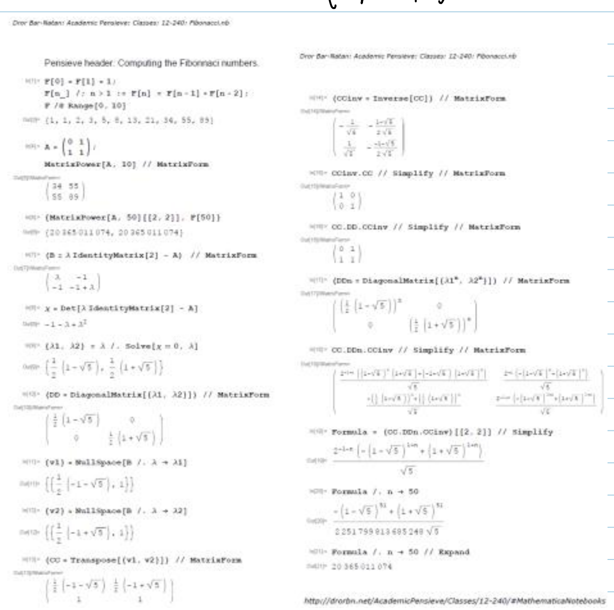
Evaluate both sides of $(*)$ on e_1, e_2, get

$$Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_2 v_2$$

$$Av = \lambda v \qquad \text{"eigenvalue, eigenvector"}$$

....

Example 2 $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n =$



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Pensieve header: Computing the Fibonacci numbers.

```
F[0] = F[1] = 1;
F[n_] /; n > 1 := F[n] = F[n - 1] + F[n - 2];
F /@ Range[0, 10]
{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89}

A = {{0, 1}, {1, 1}};
MatrixPower[A, 10] // MatrixForm
(34 55
 55 89)

{MatrixPower[A, 50][[2, 2]], F[50]}
{20 365 011 074, 20 365 011 074}

(B = λ IdentityMatrix[2] - A) // MatrixForm
(λ   -1
 -1  -1 + λ)

χ = Det[λ IdentityMatrix[2] - A]
-1 - λ + λ^2

{λ1, λ2} = λ /. Solve[χ == 0, λ]
{1/2 (1 - √5), 1/2 (1 + √5)}

(DD = DiagonalMatrix[{λ1, λ2}]) // MatrixForm
(1/2 (1 - √5)   0
 0              1/2 (1 + √5))

{v1} = NullSpace[B /. λ -> λ1]
{{1/2 (-1 - √5), 1}}

{v2} = NullSpace[B /. λ -> λ2]
{{1/2 (-1 + √5), 1}}

(CC = Transpose[{v1, v2}]) // MatrixForm
(1/2 (-1 - √5)   1/2 (-1 + √5)
 1               1)
```

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```
(CCinv = Inverse[CC]) // MatrixForm
(-1/√5   -(1 - √5)/(2 √5)
 1/√5    -(-1 - √5)/(2 √5))

CCinv.CC // Simplify // MatrixForm
(1 0
 0 1)

CC.DD.CCinv // Simplify // MatrixForm
(0 1
 1 1)

(DDn = DiagonalMatrix[{λ1^n, λ2^n}]) // MatrixForm
((1/2 (1 - √5))^n   0
 0                  (1/2 (1 + √5))^n)

CC.DDn.CCinv // Simplify // MatrixForm
[illegible]

Formula = (CC.DDn.CCinv)[[2, 2]] // Simplify
2^(-1-n) (-(1 - √5)^(1+n) + (1 + √5)^(1+n)) / √5

Formula /. n -> 50
(-(1 - √5)^51 + (1 + √5)^51) / (2 251 799 813 685 248 √5)

Formula /. n -> 50 // Expand
20 365 011 074
```

http://drorbn.net/AcademicPensieve/Classes/12-240/#MathematicaNotebooks

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Also did: Theorem If $Av_i = \lambda_i v_i$ w/ n distinct λ_i, then the v_i are

lin indep. & the matrix $C = (v_1 | \dots | v_n)$
is invertible.