240Algebral-121122, Hour 31: Determinants

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Determinants:

Mention the name "Guassian Elimination".

1. Applications: mention 2, prove 1, use none.
2. Formulas: Discuss just one.
3. Basic properties: Our core subject.

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det is a certain specific function, $\det : M_{n\times n}(F) \to F$, which we will properly define later; $\det(A) \equiv |A|$.

1. A invertible $\Longleftrightarrow \det(A) \neq 0$
2. $\left| \det \begin{pmatrix} - r_1 - \\ - r_2 - \\ \vdots \\ - r_n - \end{pmatrix} \right| = \mathrm{vol}\left(\begin{matrix}\text{Parallelopiped generated} \\ \text{by } r_1 \dots r_n\end{matrix}\right)$

$|(a_{11})| := a_{11}$ $\quad \left| \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \right| := \sum_{j=1}^{n} (-1)^{1+j} a_{1j} |A_{\hat{1}\hat{j}}|$

Examples $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ $\quad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ "Cofactor expansion"

Basic properties: 0. $\det(I) = 1$. Proof now!

1. $\det(E^1_{ij} A) = -\det(A)$ $\quad [\det E^1_{ij} = -1]$
   "Exchanging two rows flips the sign of det"
2. $\det(E^2_{i,c} A) = c \det A$ $\quad [\det E^2_{i,c} = c]$
   "multiplying a row by c multiplies det by c." [even for $c=0$!]
3. $\det(E^3_{i,j,c} A) = \det A$ $\quad [\det E^3_{i,j,c} = 1]$
   "adding c times one row to another does not change det"

Proofs later.

Thm Using these properties, the determinant of any $n\times n$ matrix A can be computed.

Pf Row reduce A keeping track of the affect on $\det A$; For r.r.e.f B, $\det B = 1$ if $B = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$, and $\det B = 0$ if B has a row of 0's.

done line

Examples $\det\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $\quad \det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

old cut line

Corollary All that there is to know about determinants can

<u>Corollary</u> All that there is to know about determinants can be deduced from 0–3; also if det′ satisfies 0–3, then det′ = det.

<u>Thm</u> A is invertible iff $\det(A) \neq 0$

<u>Thm</u> If $A = E_1 \cdots E_n$ is a product of elementary matrices, then $\det A = \det(E_1) \cdot \det(E_2) \cdots \det(E_n)$

<u>Claim</u> For square matrices, AB invertible $\Leftrightarrow$ A & B are inv.

$\Leftarrow$ $(AB)^{-1} = B^{-1}A^{-1}$

$\Rightarrow$ $B(AB)^{-1}$ is a right inverse for A, & for square matrices, if $AC = I$ then also $CA = I$.

<u>Thm</u> $\det A \cdot B = \det A \cdot \det B$

<u>Thm</u> $\det A^T = \det A$

<u>Thm</u> Everything that's true for rows is also true for columns.

Skipped extras: 1. Other formulas for det. (row/col expansions, permutations)

2. A det formula for A^{-1} & Kramer's law.

1. It's in all the books.

2. I've never used it in my life.

... recall the formula for dets & sketch the proof of the basic properties:

$$|(a_{11})| := a_{11} \qquad \left|\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}\right| := \sum_{j=1}^{n} (-1)^{1+j} a_{1j} |A_{\hat{1}\hat{j}}|$$

Then prove:

1. Linear in the first row.
2. Multilinear in the rows.
3. Vanishes if the first two rows are equal.
4. Vanishes if two adjacent rows are equal.
5. Switches sign if two adjacent rows are interchanged.
6. Switches sign whenever two rows are interchanged.
7. $E^?_{i,c}$ & $E^?_{i,j,c}$ behaviour.