240Algebral-121120, Hours 29-30: More row reduction

November-19-12
2:28 PM

HW 8 on web by midnight.

On board:

Row/col reduction:

* Interchange two rows/cols
* multiply a row/col by $c \neq 0$
* Add c times row/col j to row/col i

* Implemented by $A \mapsto EA, AE$
* Preserves ranks.
* Can reach $\left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right)$ (rank $= k$)
* Can compute inverses

How far can you go with row reduction?

(And then with cols? BTW, this is an amazing app of associativity)

1. The first non-zero entry in each row ("the pivot") is a 1.
2. In the column of a pivot, all else is 0 [Scan from left to right, to prevent interference]
3. Going down the rows, the pivots are further & further to the right.

"reduced row echelon form" r.r.e.f

Example:

$$\begin{bmatrix} 1 & 0 & 2 & 9 & 0 & e \\ 0 & 1 & -3 & 7 & 0 & \pi \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{non-zero rows} \quad \text{Pivotal rows} \\ \left.\right\} \text{zero rows / non-pivotal rows.} \end{array}$$

pivotal col's   non-pivotal col's

..... And now with col. ops., can reach $\left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right)$

Claim The rank of a r.r.e.f matrix is the number of pivots / non-zero rows in it.

Claim If A is invertible, its r.r.e.f. is I

$$\begin{cases} 2x-7y=-3 \\ 2y-x=0 \end{cases}$$

$(x,y)=(2,1)$

In this case,

$A=\begin{pmatrix} 2 & -7 \\ -1 & 2 \end{pmatrix}$ $\quad A^{-1}=\begin{pmatrix} -\frac{2}{3} & -\frac{7}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{pmatrix}$

In general

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots\dots + a_{mn}x_n = b_m \end{matrix} \Rightarrow Ax=b \text{ where } A=\begin{pmatrix} a_{11} & & a_{1n} \\ & \ddots & \\ a_{m1} & & a_{mn} \end{pmatrix}, \quad b=\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

"solving for the coordinates of an unknown vector"

Taxonomy: $Ax=0$: homogeneous system of lin. eqns

$Ax=b$: inhomogeneous system of lin eqns

If we are lucky and A is invertible, then $x=A^{-1}b$. Often we are, but often we are not.

<u>The homogeneous case</u> 1. x is a sol'n iff $x \in \ker A$. (so the set of sol'n is always a subspace of F^n)

2. 0 is always a sol'n

<u>The general case</u>

1. A sol'n exits iff $b \in R(A) = \operatorname{im}(A) = \text{col-space}(A)$.
2. If x_0 is a sol'n then x_1 is also a sol'n iff $x_1 = x_0 + x$ where x is a sol'n of the homogeneous eq'n $Ax=0$ ("affine subspace")

$Ax=b \Leftrightarrow \underbrace{EAx}_{Cx} = \underbrace{Eb}_{d}$ $\qquad (A \mid b) \xrightarrow[\text{ops}]{\text{row}} (C \mid d)$

If C is r.r.e.f.:

Example: $\left[\begin{array}{cccccc|c} 1 & 0 & 2 & 9 & 0 & e & d_1 \\ 0 & 1 & -3 & 7 & 0 & \pi & d_2 \\ 0 & 0 & 0 & 0 & 1 & 2 & d_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_4 \end{array}\right]$ $\begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{non-zero rows} \quad \text{Pivotal rows} \\ \} \text{ zero rows / non-pivotal rows.} \end{matrix}$

pivotal $\qquad$ non-pivotal

$$\left[\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & d_4 \end{array}\right] \Big\} \text{ zero rows / non-pivotal rows.}$$

pivotal col's

non-pivotal col's

1. Sol'n exist iff the d_i's in the non-pivotal rows are 0.
2. The x_j's corresponding to the non-pivotal col's can be set arbitrarily, the x_j's corresponding to the pivotal rows are then fixed.

Example

$$\begin{aligned} 2x_1 + 3x_2 + x_3 + 4x_4 - 9x_5 &= 17 \\ x_1 + x_2 + x_3 + x_4 - 3x_5 &= 6 \\ x_1 + x_2 + x_3 + 2x_4 - 5x_5 &= 8 \\ 2x_1 + 2x_2 + 2x_3 + 3x_4 - 8x_5 &= 14, \end{aligned}$$

$$\left(\begin{array}{ccccc|c} 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & -1 & 4 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

and so...

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2t_1 + 2t_2 + 3 \\ t_1 - t_2 + 1 \\ t_1 \\ 2t_2 + 2 \\ t_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$