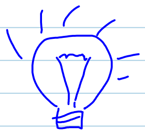
240Algebral-121115, Hour 28: Row and Column Reduction

November-14-12
2:19 PM

HW7 is on web.

Comment 2 $\operatorname{rank} A = \operatorname{rank} PAQ$ whenever $P \in M_{m\times m}$ & $Q \in M_{n\times n}$ are invertible.



Look for P & Q that will make PAQ "simpler" than A.

Q1 Which P, Q? Q2 What's simpler?

Ans 2 $\operatorname{rank}\begin{pmatrix} \overbrace{1 \; \ddots \; 1}^{k} & 0 \\ 0 & 0 \end{pmatrix} = \operatorname{rank}\left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right) = k$

Ans 1 Examples of "good" P/Q: "elementary matrices"

1. Interchanging rows/columns. $E^1_{i,j}$ $(r_i \leftrightarrow r_j)$ or $(c_i \leftrightarrow c_j)$
2. Multiplying r/c by a scalar. $E^2_{i,c}$ $(r_i *= c)$ or $(c_i *= c)$
3. Adding a multiple of one r/c to another. $E^3_{i,j,c}$ $(r_i += c\, r_j)$, ...

"row/column reduction"

Added Dec 17: I should have used this notation.

Thm Every matrix A can be r/c-reduced to a block matrix of the form $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$.

Problem. Find the rank the matrix

$$A = \begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}.$$

Solution. Using (invertible!) row/column operations we aim to bring A to look as close as possible to an identity matrix:

Do	Get	Do	Get
1. Bring a 1 to the upper left corner by swapping the first two rows and multiplying the first row (after the swap) by $1/4$.	$\begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$	2. Add (-8) times the first row to the third row, in order to cancel the 8 in position 3-1.	$\begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$
3. Likewise add (-6) times the first row to the fourth row, in order to cancel the 6 in position 4-1.	$\begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix}$	4. With similar column operations (you need three of those) cancel all the entries in the first row (except, of course, the first, which is used in the canceling).	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix}$
5. Turn the 2-2 entry to a 1 by multiplying the second row by $1/2$.	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix}$	6. Using two row operations "clean" the second column; that is, cancel all entries in it other than the "pivot" 1 at position 2-2.	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix}$
7. Using three column	$(1 \;\; 0 \;\; 0 \;\; 0 \;\; 0)$	8. Clean up the row and the column of the 4 in position	$(1 \;\; 0 \;\; 0 \;\; 0 \;\; 0)$

	$\begin{pmatrix} 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix}$
7. Using three column operations clean the second row except the pivot.	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix}$	8. Clean up the row and the column of the 4 in position 3-3 by first multiplying the third row by $1/4$ and then performing the appropriate row and column transformations. Notice that by pure luck, the 4 at position 4-5 of the matrix gets killed in action.	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Thus the rank of our matrix is 3.

http://drorbn.net/index.php?title=12-240/Classnotes_for_Tuesday_November_8

Claim rank $A = \text{rank}(A^T)$ BTW, the meaning of A^T in the world of l.t. is quite intricate.

claim rank $A = \dim(\text{col-space}(A)) = \dim(\text{row-space}(A))$

done in tutorials

Suppose you could row reduce A to I. Find A^{-1}.

$$E_4E_3E_2E_1A = I \Rightarrow A^{-1} = E_4E_3E_2E_1$$

* The hard way.
* The easy way: r.r. $(A \mid I)$

Example: compute $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1}$.