Next goals: 1. Compute rank T/A. 2. Compute A^{-1} (when possible)
3. Solve systems of linear eqn's.

Proposition Given $V' \xrightarrow{Q} V \xrightarrow{T} W \xrightarrow{P} W'$ with invertible P&Q, rank T = rank PTQ [enough that Q surjective & P injective]

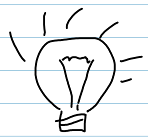
PF
$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[d, "P"] & \supset \operatorname{im}(T) = C, \quad \text{basis} = (w_i = T(v_i))_{i=1}^{r} \\ V' \arrow[u, "Q"] \arrow[r, "T'"', "PTQ"] & W' & \supset \operatorname{im}(T') = C' \quad \text{basis} = (w_i' = P(w_i))_{i=1}^{r} \end{tikzcd}$$

Need: 1. $w_i' \in \operatorname{im} T'$; meaning $\exists v_i' \in V'$ s.t. $w_i' = T'v_i'$
2. w_i' span C'
3. w_i' are lin. indep.

Def If $A \in M_{m\times n}$, let rank $A :=$ rank T_A, where T_A is the "standard" $T_A : F^n \to F^m$

Comment 1 rank $[T]_\beta^\gamma$ = rank T PF.
$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\,]_\beta}"'] & W \arrow[d, "{[\,]_\gamma}"] \\ F^n \arrow[r, "A"', "T_A"] & F_m \end{tikzcd}$$

Comment 2 rank A = rank PAQ whenever $P \in M_{m\times m}$ & $Q \in M_{n\times n}$ are invertible.



Look for P & Q that will make PAQ "simpler" than A.

done line

Q1 which P, Q?

Q2 what's simpler?

Ans 2 rank $\begin{pmatrix} \overbrace{1 \; \ddots \; 1}^{k} & 0 \\ 0 & 0 \end{pmatrix} = \text{rank} \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right) = k$

Ans 1 Examples of "good" P/Q: "elementary matrices"

1. Interchanging rows/columns. $E^1_{i,j}$ $(\underrightarrow{r_i \leftrightarrow r_j})$ or $(\underrightarrow{c_i \leftrightarrow c_j})$
2. Multiplying r/c by a scalar. $E^2_{i,c}$ $(\underrightarrow{r_i *= c})$ or $(\underrightarrow{c_i *= c})$
3. Adding a multiple of one r/c to another. $E^3_{i,j,c}$ $(r_i += c\, r_j)$, ...

"row/column reduction"

Added Dec 7: I should ...

Thm Every matrix A can be r/c-reduced to a

Thm Every matrix A can be r/c-reduced to a block matrix of the form $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$.

Added Dec 14: I should have used this notation

Problem. Find the rank the matrix

$$A = \begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}.$$

Solution. Using (invertible!) row/column operations we aim to bring A to look as close as possible to an identity matrix:

Do	Get	Do	Get
1. Bring a 1 to the upper left corner by swapping the first two rows and multiplying the first row (after the swap) by $1/4$.	$\begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$	2. Add (-8) times the first row to the third row, in order to cancel the 8 in position 3-1.	$\begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$
3. Likewise add (-6) times the first row to the fourth row, in order to cancel the 6 in position 4-1.	$\begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix}$	4. With similar column operations (you need three of those) cancel all the entries in the first row (except, of course, the first, which is used in the canceling).	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix}$
5. Turn the 2-2 entry to a 1 by multiplying the second row by $1/2$.	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix}$	6. Using two row operations "clean" the second column; that is, cancel all entries in it other than the "pivot" 1 at position 2-2.	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix}$
7. Using three column operations clean the second row except the pivot.	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix}$	8. Clean up the row and the column of the 4 in position 3-3 by first multiplying the third row by $1/4$ and then performing the appropriate row and column transformations. Notice that by pure luck, the 4 at position 4-5 of the matrix gets killed in action.	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Thus the rank of our matrix is 3.

http://drorbn.net/index.php?title=12-240/Classnotes_for_Tuesday_November_8