HW5 Due!

Peter Evan RS211

Brandon odd RW110

V/F, basis $\beta = (v_1 \dots v_n)$ $\quad W/F$, basis $\gamma = (w_1 \dots w_m)$

Abstract, general, coord-free $\qquad$ merely numbers, choice-dependent, easy to work with

$$\mathcal{L}(V, W) \longrightarrow M_{m\times n}(F)$$

$$T \longrightarrow [T]^\gamma_\beta = A$$

$$A = \begin{pmatrix} a_{11} & & & a_{1n} \\ [Tv_1]_\gamma & [Tv_2]_\gamma & \dots & [Tv_n]_\gamma \\ a_{m1} & & & a_{mn} \end{pmatrix} \Longleftrightarrow Tv_j = \sum_{i=1}^{m} a_{kj} w_k$$

\begin{tikzcd}
U \arrow[r, "S"', "B"] \arrow[rr, bend right, "T\circ S"', "C"] & V \arrow[r, "T"', "A"] & W
\end{tikzcd}

basis: α, β, γ; dim: p, n, m

If you know A and B, can you find C?

⋮

derive C, then:

<u>Definition</u> $A \in M_{m\times n}$, $B \in M_{n\times p}$ $\quad A\cdot B \in M_{m\times p}$ by $(A\cdot B)_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}$

<u>Example</u> $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} = \dots$

I should have had another example, $T_A v = Av$ where v is regarded as an $n\times 1$ matrix

<u>Thm</u> $[T\circ S]^\gamma_\alpha = [T]^\gamma_\beta \cdot [S]^\beta_\alpha$

<u>Example</u> $T_\beta \circ T_\alpha = T_{\alpha+\beta}$ for rotations.

The good and the bad about "matrix algebra":

Good	Bad
1. $A+B = B+A$, $(A+B)+C = A+(B+C)$ (basically, all works for addition)	1. Addition is defined only for matrices of same dims.
2. $A(B\cdot C) = (A\cdot B)C$ $\exists I$ s.t. $A\cdot I = A$, $IA = A$	2. mult. is defined only if "mid" dimension matches & produces an output of yet other dims.
3. If $A\cdot A^{-1} = I$, then $A^{-1}A = I$	3. A^{-1} may not exist even if $A \neq 0$.
4. $(A+B)C = AC + BC$ $A(B+C) = AB + AC$	4. Generally, $A\cdot B \neq B\cdot A$, even when both make sense.

done

line

Next goals: 1. Compute rank T/A. 2. Compute A^{-1} (when possible) 3. Solve systems of linear eqn's. } line stated

---

<u>Proposition</u> Given $V' \xrightarrow{Q} V \xrightarrow{T} W \xrightarrow{P} W'$ with invertible P & Q, rank T = rank PTQ [enough that Q surjective & P injective]

<u>Pf</u>

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, phantom] & W \arrow[d, "P"] \\ V' \arrow[u, "Q"] \arrow[r, "T' = PTQ"'] & W' \end{tikzcd}$$

$W \supset \mathrm{im}(T) = C$, basis $= (w_i = T(v_i))_{i=1}^{r}$

$\Downarrow$

$W' \supset \mathrm{im}(T') = C'$ basis $= (w_i' = P(w_i))_{i=1}^{r}$

Need: 1. $w_i' \in \mathrm{im}\, T'$; meaning $\exists v_i' \in V'$ s.t. $w_i' = T' v_i'$

2. w_i' span C'

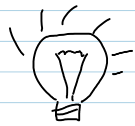
3. w_i' are lin. indep.

<u>Def</u> If $A \in M_{m\times n}$, let rank $A :=$ rank T_A, where T_A is the "standard" $T_A : F^n \to F^m$

<u>Comment 1</u> rank $[T]_\beta^\gamma$ = rank T <u>Pf.</u>

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\ ]_\beta}"'] & W \arrow[d, "{[\ ]_\gamma}"] \\ F^n \arrow[r, "A", "T_A"'] & F^m \end{tikzcd}$$

<u>Comment 2</u> rank A = rank PAQ whenever $P \in M_{m\times m}$ & $Q \in M_{n\times n}$ are invertible.



Look for P & Q that will make PAQ "simpler" than A.

<u>claim</u> rank $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} = k$

Examples of "good" P/Q: 1. Interchanging rows/columns.

2. Multiplying r/c by a scalar.

3. Adding a multiple of one r/c to another.