240AlgebraI-121030, Hours 22-23: Rank-Nullity

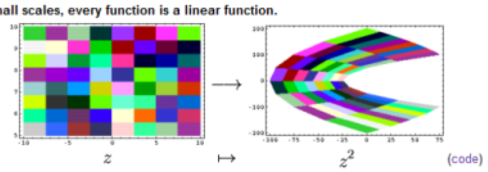
October-25-12
1:36 PM

HW5 is on web!
TT results:

Show the QuiltPlot handout!

Linear Algebra is the Small Scale Theory of *Everything*

- To study the large, start with the small.
- In small scales, every space is a vector space.
  - Indeed if you walk a mile east, a mile north, a mile west and a mile south, you're back were you started, but if you fly a 1,000 miles east, a 1,000 miles north, a 1,000 miles west and a 1,000 miles south, you're not back were you started (where will you be?).
- In small scales, every function is a linear function.



- The world doesn't come with coordinates.
  - Hence whenever we can we work without a basis, and when we do study bases, we study all of them.

Fix a l.f. $T: V \to W$

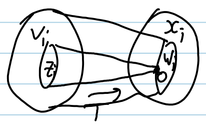
Def $N(T) = \ker T = \{v : Tv = 0\}$ "null space", "kernel".
$R(T) = \operatorname{im} T = \{Tv : v \in V\}$ "range", "image"

Prop/Def $N(T) \subset V$ is a subspace; $\text{nullity}(T) := \dim N(T)$
$R(T) \subset W$ is a subspace; $\text{rank}(T) := \dim R(T)$

Examples 0, I_V, $D: P_n(\mathbb{R}) \to P_n(\mathbb{R})$

Thm1 "the dimension theorem", "the rank-nullity thm"
Given $T: V \to W$, $\underset{m}{\dim V} = \underset{r}{\text{rank}(T)} + \underset{n}{\text{nullity}(T)}$

Pf $(z_i)_1^n$ basis of $N(T)$, extend to $(z_i) \cup (v_i)$ a basis of V,



claim $w_i := T(v_i)$ are lin. indep. in W pf....
claim w_i span $R(T)$ pf...

Corollary of Thm 1 If $\dim V = \dim W$, TFAE

1. T is 1-1
2. T is onto
3. $\text{rank}\, T = \dim V$
4. T is invertible.

Thm 2 $T: V \to W$ & $T': V' \to W'$ are "isomorphic" iff $(\dim V, \dim W, \text{rank}\, T) = (\dim V', \dim W', \text{rank}\, T')$

Skipable.

i.e., $\exists$ a "commutative square of isomorphisms":

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi"'] & W \arrow[d, "\psi"] \\ V' \arrow[r, "T'"] & W' \end{tikzcd}$$

Reminder: Choosing a basis, V is isomorphic to F^n.

Goal: 1. The set $\mathcal{L}(V, W)$ of all lin. trans. $V \to W$ is a vector space.

2. Choosing bases, it is isomorphic to $M_{m \times n}$
$(m = \dim W,\ n = \dim V)$

Then follow October 26, 2006:

Let $\beta = (u_1, \ldots, u_n)$ be an ordered basis of a f.d. V.S. V. If $x = \sum a_i u_i$, write

$$[x]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \left(= Tx, \text{ if } T \text{ is the iso. } V \to F^n \text{ given by } u_i \to e_i \right)$$

The coords of x rel. to β.

Example in $P_2(\mathbb{R})$ $\quad [x^2 - 2x + 3]_{(1, x, x^2)} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

done line

Def Given $T: V \to W$ a lin trans. and ordered bases $\beta = (v_1, \ldots v_n)$ of V & $\gamma = (w_1, \ldots w_m)$ of W,

Let $\quad A = [T]_\beta^\gamma = \left([Tv_1]_\gamma \middle| [Tv_2]_\gamma \middle| \cdots \middle| [Tv_n]_\gamma \right) \in M_{m \times n}(F)$

Note 1. T can be reconstructed from $[T]_\beta^\gamma$

2. Every matrix arises in this way.

Examples ~~2~~ 1. $D: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ by differentiation.
$\quad$ 2:0 $\quad$ 1:1
3. T_θ

Def $\mathcal{L}(V, W)$

Claim 1. $\mathcal{L}(V, W)$ is a V.S.

2. $T \longmapsto [T]_\beta^\gamma$ is an isomorphism of V.S.

$$\mathcal{L}(V, W) \longrightarrow M_{m \times n}(F)$$