240Algebra1-121011, Hour 15: Corollaries of replacement

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7:54 PM

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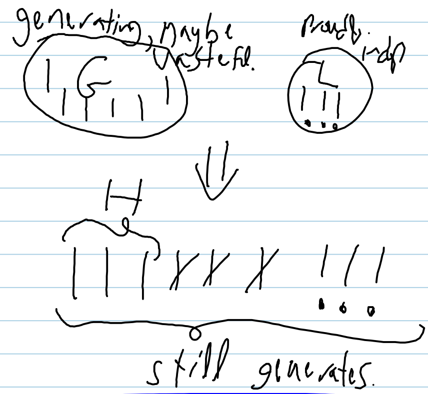
Riddle Along. (Thanks, ...) Five piles of 100-gram gold coins are given, but it is known that the coins in one of the piles are fakes, and weigh only 99 grams. Find which pile it is, with only one use of an accurate scale.

TT discussion: next class.

<u>Lemma</u> (the replacement lemma)
$|G|=n$, $\operatorname{span} G = V$, L lin indep
$\Rightarrow |L| \le n$ & $\exists H \subset G$ with
$|H| = n - |L|$ and $\operatorname{span}(H \cup L) = V$



start line

Corollaries: 1. If V has a finite basis β_1 then every other basis β_2 of V is also finite & $|\beta_1| = |\beta_2|$.

2. "$\dim V$" makes sense.

3. Assume $\dim V = n$. Then

a. If G generates V, $|G| \ge n$ & a subset of G is a basis.
if also $|G| = n$, then G itself is a basis.

b. If L is linearly indep in V, then $|L| \le n$;
if also $|L| = n$, L is a basis.
if also $|L| < n$, L can be extended to a basis.

4. If V is finite-dimensional and $W \subset V$ is a subspace, then W is f.d. and $\dim W \le \dim V$. } done minus "W is f.d."

If also $\dim W = \dim V$, then $W = V$.

If also $\dim W < \dim V$, then any basis of W . . . not done can be extended to a basis of V.

The Lagrange interpolation formula:

Let x_i be distinct pts in $\mathbb{R}/\mathbb{F}$ $\quad i=1,\dots,n+1$

Let y_i be any pts in $\mathbb{R}/\mathbb{F}$.

Q Can you find a polynomial $P \in P_n(\mathbb{R})$ s.t. $P(x_i)=y_i$?

Is it unique?

Who cares? * Scientists.

* Computer drawing programs.

Follow through w/ example.

$p(0)=5$, $p(1)=2$, $p(3)=2$

$P_1 = \frac{(x-1)(x-3)}{3} = \frac{1}{3}(x^2-4x+3)$

$P_2 = \frac{x(x-3)}{-2} =$

$P_3 = \frac{x(x-1)}{6} = \dots$

$p = x^2-4x+5$

Solution Let $\tilde{P}_i(x) = \prod_{j \neq i}(x-x_j)$

Then $P_i(x_j) = \begin{cases} 0 & j \neq i \\ \neq 0 & i=j \end{cases}$

set $P_i(x) = \tilde{P}_i(x)/\tilde{P}_i(x_i) = \dots$

Then * $P(x) := \sum y_i P_i(x)$ satisfies $P(x_i)=y_i$

* $\beta = \{P_1 \dots P_{n+1}\}$ is lin. indep.
* $\Rightarrow \beta$ is a basis
* Every $f \in P_n(\mathbb{R})$ can be expressed as a lin. comb. of the P_i in a unique way.
* If $q(x)$ also satisfies $q(x_i)=y_i$, then $q(x)=P(x)$.
* Therefore the solution to our problem is unique
* Aside: If $\forall i$ $P(x_i)=0$, then $P=0$

(so a non-zero polynomial of degree n has at most n roots.)