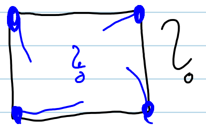
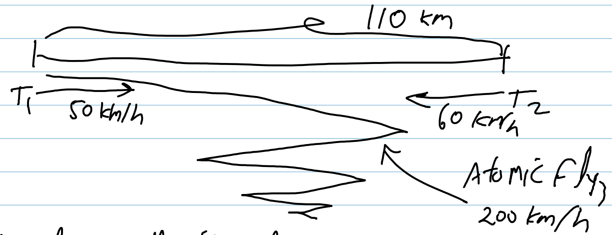
HW 4 on web!

Riddle Along.





Thu: A better fly bouncing problem | How long will it fly befor crashing?

Def Basis $\beta \subset V$

Thm A subset $\beta \subset V$ is a basis iff every $v \in V$ can be expressed in a unique way as a l.c. of elements of β

Thm If a finite set S generates a v.s. V, then there is a subset $\beta \subseteq S$ which is a basis of V

Pf Let β be a lin indep subset of S which is of maximal size. Then every $v \in S \setminus \beta$ satisfies $v \in \operatorname{span}\beta$, so $S \subset \operatorname{span}\beta$, so $\operatorname{span} S \subset \operatorname{span}\beta$.

Lemma If S is lin indep in V and $v \in V \setminus S$, then $S \cup \{v\}$ is lin. dep. iff $v \in \operatorname{span}(S)$.

Our first non-language theorem:

Thm If a v.s. V has a finite basis, then every other basis of V has the same number of elements in it.

Def If V has a finite basis, we say that it is "finite-dimensional" and let

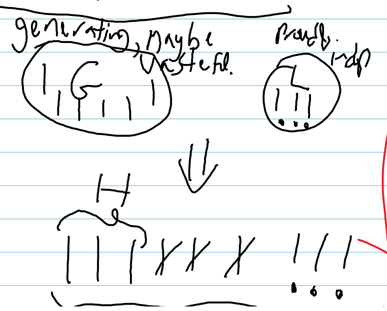
$$\dim V := \begin{pmatrix} \text{The number of elements} \\ \text{in (any) basis of } V \end{pmatrix}$$

Examples as before: $\{0\}$, F^n, $M_{m\times n}$, $P_n(F)$, $P(F)$

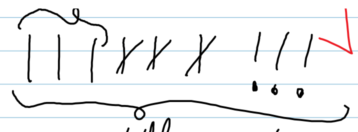
Lemma (the replacement lemma)

$|G| = n$, $\operatorname{span} G = V$, L lin indep

$\Rightarrow |L| \leq n$ & $\exists H \subset G$ with $|H| = n - |L|$ and $\operatorname{span}(H \cup L) = V$



Added Dec 13, 2012: I should have proven replacement without assuming the finiteness of $|G|$.

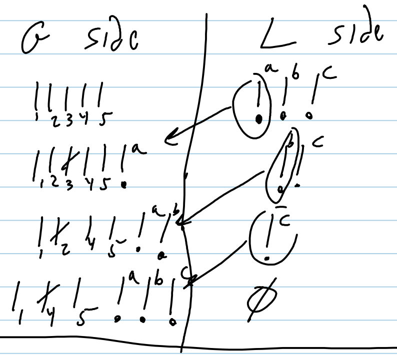
$|H| = n - |L|$ and $\mathrm{span}(H \cup L) = V$  $|G|$.

still generates.

Pf of Theorem from lemma.

Informal proof of Lemma First of all, if $\sum a_i u_i = 0$, the any vector that appears in this dependency with non-zero coeff is a l.c. of the others.

Now:



Formal proof: Induction on $|L|$. $|L|=0$: trivial.

Now $|L| = m+1$; $L = \{v_1 \ldots v_{m+1}\}$. Use $L' = \{v_1 \ldots v_m\}$, Find $H' = \{u_1 \ldots u_{n-m}\} \subset G$ s.t. $\{u_1 \ldots u_{n-m}, v_1 \ldots v_m\}$ spans. write

$$v_{m+1} = a_1 u_1 + \ldots + a_{n-m} u_{n-m} + b_1 v_1 + \ldots + b_m v_m$$

$\therefore$ Not all $a_i = 0$, so $n-m>0$, so $m+1 \le n$.

$\therefore$ w.l.o.g. $a_1 \neq 0$, so $u_1 \in \mathrm{span}(u_2, \ldots u_{n-m}, v_1 \ldots v_{m+1})$, so take $H = \{u_2 \ldots u_{n-m}\}$.

done line

Corollaries: 1. If V has a finite basis β_1 then every other basis β_2 of V is also finite & $|\beta_1| = |\beta_2|$.

2. "dim V" makes sense.

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3. Assume $\dim V = n$. Then

a. If G generates V, $|G| \ge n$ & if also $|G| = n$, then G is a basis.

b. If L is linearly indep in V, the $|L| \le n$;

if also $|L| = n$, L is a basis.

if also $|L| < n$, L can be extended to a basis.

4. If V is finite-dimensional and $W \subset V$ is a subspace, then W is f.d. and $\dim W \le \dim V$.

If also $\dim W = \dim V$, then $W = V$.

If also $\dim W < \dim V$, then any basis of W can be extended to a basis of V.