$V_L = V_S$

Riddle Along.
Read Along. 1.4–1.6.
Web Fact: (not visible) = (doesn't exist).
Life Fact. No "teaching over email".
Reminders. We seek "basis"; l.c.; span; "generates"

Def A subset $S \subset V$ is "lin. dep" if it is "wasteful".
I.e., If $\exists\, a_i \in F$ not all 0 & $u_i \in S$ s.t. $\sum a_i u_i = 0$.
Otherwise, it is "lin. indep."

Examples $\{e_i\}$ ✓, ~~$\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$~~

Comments 1. $\emptyset$ is lin. indep.
2. $\{u\}$ is lin indep iff $u \neq 0$.
3. Suppose $S_1 \subset S_2 \subset V$. Then
   a. If S_1 is dep, so is S_2
   b. If S_2 is indep, so is S_1
4. If S is lin indep in V and $v \in V \setminus S$, then $S \cup \{v\}$ is lin. dep. iff $v \in \operatorname{span}(S)$. } skipped

Def Basis $\beta \subset V$

Examples: 1. $\emptyset$ for $\{0\}$.
2. e_i for F^n 3. E^{ij} for $M_{m\times n}(F)$
4. $(1, x, \ldots x^n)$ for $P_n(F)$
5. $(1, x, \ldots\quad)$ for $P(F)$
6. $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$ for $\mathbb{R}^2$. $\begin{pmatrix}a\\b\end{pmatrix} = \frac{a+b}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{a-b}{2}\begin{pmatrix}1\\-1\end{pmatrix}$

done line

Thm A subset $\beta \subset V$ is a basis iff every $v \in V$ can be expressed in a unique way as a l.c. of elements of β.

Thm If a finite set S generates a v.s. V, then there is a subset $\beta \subseteq S$ which is a basis of V

Pf Let β be a lin indep subset of S which is of maximal size. The every $v \in S \setminus \beta$ satisfies $v \in \operatorname{span}\beta$, so $S \subset \operatorname{span}\beta$, so $\operatorname{span} S \subset \operatorname{span}\beta$.

Our first non-language theorem:

Thm If a v.s. V has a finite basis, then every other basis of V has the same number of elements in it.

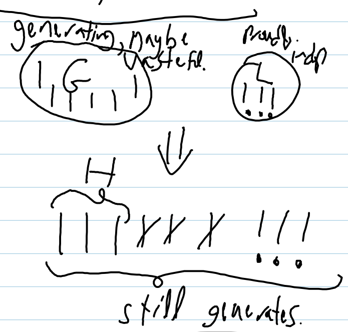
target line

Def If V has a finite basis, we say that it is "finite-dimensional" and let

$$\dim V := \begin{pmatrix} \text{The number of elements} \\ \text{in (any) basis of } V \end{pmatrix}$$

Examples as before:
$\{0\}$, F^n, $M_{m \times n}$, $P_n(F)$, $P(F)$

Lemma (the replacement lemma)
$|G| = n$, $\operatorname{span} G = V$, L lin indep
$\Rightarrow |L| \le n$ & $\exists H \subset G$ with
$|H| = n - |L|$ and $\operatorname{span}(H \cup L) = V$

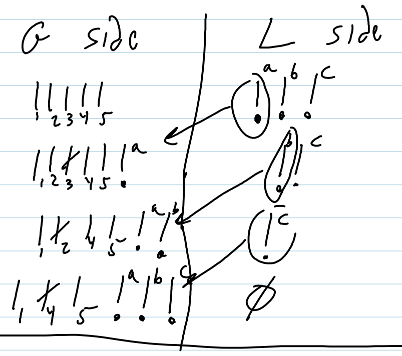


PF of theorem from lemma.

Informal proof of Lemma First of all, if $\sum a_i u_i = 0$, the any vector that appears in this dependency with non-zero coeff is a l.c. of the others.

09 hour 13

Now:



Formal proof: Induction on $|L|$. $|L| = 0$: trivial.

Now $|L| = m+1$; $L = \{v_1 \ldots v_{m+1}\}$. Use $L' = \{v_1 \ldots v_m\}$, Find $H' = \{u_1 \ldots u_{n-m}\} \subset G$ s.t. $\{u_1 \ldots u_{n-m}, v_1 \ldots v_m\}$ spans. write

$$v_{m+1} = a_1 u_1 + \ldots + a_{n-m} u_{n-m} + b_1 v_1 + \ldots + b_m v_m$$

$\therefore$ Not all $a_i = 0$, so $n - m > 0$, so $m+1 \le n$.

$\therefore$ w.l.o.g. $a_1 \neq 0$, so $u_1 \in span(u_2, \ldots u_{n-m}, v_1 \ldots v_{m+1})$,

so take $H = \{u_2 \ldots u_{n-m}\}$.

Corollaries: 1. If V has a finite basis β_1 then every other basis β_2 of V is also finite & $|\beta_1| = |\beta_2|$.

2. "dim V" makes sense.

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3. Assume $\dim V = n$. Then

   a. If G generates V, $|G| \geq n$ & if also $|G| = n$, then G is a basis.

   b. If L is linearly indep in V, the $|L| \leq n$;
      if also $|L| = n$, L is a basis.
      if also $|L| < n$, L can be extended to a basis.

4. If V is finite-dimensional and $W \subset V$ is a subspace, then W is f.d. and $\dim W \leq \dim V$.
   If also $\dim W = \dim V$, then $W = V$.
   If also $\dim W < \dim V$, then any basis of W can be extended to a basis of V.