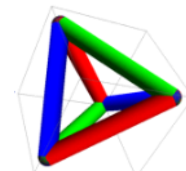
The Simplicity of the Alternating Groups

Suggestion for a good deed: TeX this up nicely!

This handout is to be read twice: first read red only, to ascertain that everything in red is easy and boring, then read black and red, to actually understand the proof.

Theorem. The alternating group $A_n \triangleleft S_n$ is simple for $n \neq 4$.

Remark. Easy for $n \leq 3$, false for $n=4$ as there is $\phi: A_4 \twoheadrightarrow A_3$, so assume $n \geq 5$.



Lemma 1. Every element of A_n is a product of 3-cycles.

Pf. Every $\sigma \in A_n$ is a product of an even number of 2-cycles, and $(12)(23)=(123)$ & $(123)(234)=(12)(34)$.

Lemma 2. If $N \triangleleft A_n$ contains a 3-cycle, then $N = A_n$.

Pf. WLOG, $(123) \in N$. Then for all $\sigma \in S_n$, $(123)^\sigma \in N$: if $\sigma \in A_n$, this is clear. Otherwise $\sigma = (12)\sigma'$ w/ $\sigma' \in A_n$, and then as $(123)^{(12)} = (123)^2$, $(123)^\sigma = ((123)^2)^{\sigma'} \in N$. So N contains all 3-cycles.

Case 1. N contains an element w/ cycle of length ≥ 4.

Resolution. $\sigma = (123456)\sigma' \in N \Rightarrow \sigma^{-1}(123)\sigma(123)^{-1} = (136) \in N$

Case 2. N contains an element w/ 2 cycles of length 3.

Res. $\sigma = (123)(456)\sigma' \in N \Rightarrow \sigma^{-1}(124)\sigma(124)^{-1} = (14263) \in N$.

Case 3. N contains $\sigma = (123)\cdot$(a product of disjoint 2-cycles).

Res. $\sigma^2 = (132) \in N$

Case 4. Every element of N is product of disjoint 2-cycles.

Res. $\sigma = (12)(34)\sigma' \Rightarrow \sigma^{-1}(123)\sigma(123)^{-1} = (13)(24) = \tau \in N$
$\Rightarrow \tau^{-1}(125)\tau(125)^{-1} = (13452) \in N$ $\square$