

September-25-11
11:58 PM

- On board.
1. Class photo at 10:55!
 2. HW1 is on web!
 3. $x^g = g^{-1} x g$ so $(x^g)^h = x^{gh}$ (For Selick: $(x^g)^h = x^{hg}$)
 4. If $\sigma, \tau \in S_n$, then $\sigma \tau = \sigma \circ \tau$!
 5. Today's Agenda: 1. Jordan Hölder.
2. A_n is simple.
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Go over the "Selick" handout;

Example: 1. $\phi: S_4 \rightarrow S_3$

2. Is there a normal subgroup of S_4 which is isomorphic to S_3 ?

The Jordan-Hölder Theorem. Let G be a finite group. Then there exist a sequence

$G = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \dots \triangleleft G_n = \{e\}$ s.t. $H_i = G_i / G_{i-1}$ is simple. Furthermore, the sequence (H_i) , the "composition series" of G , is unique up to a permutation.

Example $S_4 \triangleleft A_4 \triangleleft \begin{pmatrix} (12)(34) \\ (13)(24) \\ (14)(23) \end{pmatrix} \triangleleft \begin{pmatrix} (12)(34) \\ \{e\} \end{pmatrix}$

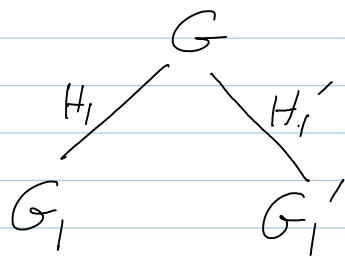
$4 \rightarrow A_4 \rightarrow A_3$
 $24 \quad 12 \quad 4 \quad 2 \quad 12 \quad 3$

Proof by induction on $|G|$.

Existence: Let G_1 be a maximal normal

Subgroup.

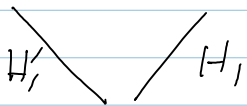
Uniqueness: Use the "Diamond Principle":



$$G \triangleright G_1 \triangleright G_2 \dots$$

$$G \triangleright G_1' \triangleright G_2' \dots$$

Claim $G = G_1 G_1'$



PF $G_1 G_1'$ is normal in G yet

bigger than G_1, G_1' .

Theorem. A_n is simple for $n \neq 4$. [Proof as in Lang's]

Cycle Decomposition. $(12)(345) = [21453] = 21453$

Claim If $\sigma = (a_1 \dots a_k)$ and $\tau = [\tau_1 \tau_2 \dots \tau_n]$,

then

$$\sigma^\tau = \tau^{-1} \sigma \tau = (\tau^{-1} a_1, \tau^{-1} a_2, \dots)$$

Corollary σ is conjugate to σ' iff they have the same cycle lengths

Corollary # (conjugacy classes of S_n) = $P(n)$

Lemma 1. Every element of A_n is a product of 3-cycles. done line

PF $(12)(23) = (123), (123)(234) = (12)(34) \dots$

Lemma 2. If $N \triangleleft A_n$ contains a 3-cycle, then $N = A_n$

PF WLOG, $(123) \in N$. Claim For $\sigma \in S_n$, $(123)^\sigma \in N$ ($\sigma \in A_n \checkmark$, $\sigma = (12)\sigma \checkmark$)

so N contains all 3-cycles... \square

Now take $N \triangleleft A_n$ w/ $N \neq \{1\}$

Case 1. N contains an element w/ cycle of length ≥ 4

$$\sigma = (123456) \sigma^{-1} \in N \quad \sigma^{-1}(123)\sigma(123)^{-1} = (136)$$

Case 2. N contains an element $\sigma = (123)(456) \sigma^{-1}$

$$\text{consider } \sigma^{-1}(124)\sigma(124)^{-1} = (14263)$$

Case 3. N contains $\sigma = (123)$ (product of pair)

$$\text{Then } \sigma^2 = (132) \dots$$

Case 4. Every element of N is a product of disjoint 2-cycles.

$$\sigma = (12)(34) \sigma^{-1} \Rightarrow \sigma^{-1}(123)\sigma(123)^{-1} = (13)(24) = \tau \in N$$

$$\Rightarrow \tau^{-1}(125)\tau(125)^{-1} = (13452) \in N$$