September 27, hours 7-8: Isomorphism Theorems review, Jordan-Holder, Simplicity of A_n

September-25-11
11:58 PM

On board. 1. Class photo at 10:55!

2. HW1 is on web!

3. $x^g = g^{-1}xg$ so $(x^g)^h = x^{gh}$ (For Selick: $(x^g)^h = x^{(hg)}$)

4. If $\sigma, \tau \in S_n$, then $\sigma\tau = \sigma \circ \tau$!

5. Today's Agenda: 1, Jordan Hölder.

   2. A_n is simple.

Go over the "Selick" handout;

Example: 1. $\phi : S_4 \to S_3$

2. Is there a normal subgroup of S_4 which is isomorphic to S_3?

The Jordan-Hölder Theorem. Let G be a finite group. Then there exist a sequence

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \dots \triangleright G_n = \{e\} \text{ s.t. } H_i = G_i / G_{i-1}$$

is simple. Furthermore, the sequence (H_i), the "composition series" of G, is unique up to a permutation.

$4 \to A_4 \to A_3$
$\quad 12 \quad\ 3$

Example $\underset{24}{S_4} \triangleright \underset{12}{A_4} \triangleright \underset{4}{\begin{matrix}(12)(34)\\(13)(24)\\(14)(23)\end{matrix}} \triangleright \underset{2}{(12)(34)} \triangleright \{e\}$

Proof by induction on $|G|$.

Existance: Let G_1 be a maximal normal

Subgroup.

Uniqueness: Use the "diamond principle":

$$\begin{tikzcd}
& G \arrow[dl, dash, "H_1"'] \arrow[dr, dash, "H_1'"] & \\
G_1 \arrow[dr, dash, "H_1'"'] & & G_1' \arrow[dl, dash, "H_1"] \\
& G_1 \cap G_1' &
\end{tikzcd}$$

$G \rhd G_1 \rhd G_2 \ldots$

$G \rhd G_1' \rhd G_2' \ldots$

Claim $G = G_1 G_1'$

Pf $G_1 G_1'$ is normal in G yet bigger that G_1, G_1'.

Theorem. A_n is simple for $n \neq 4$. [Proof as in Lang's]

Cycle Decomposition. $(12)(345) = [21453] = 21453$

Claim If $\sigma = (a_1 \ldots a_k)$ and $\tau = [\tau_1 \tau_2 \ldots \tau_n]$, then

$$\sigma^\tau = \tau^{-1} \sigma \tau = (\tau^{-1}(a_1), \tau^{-1} a_2, \ldots \quad)$$

Corollary σ is conjugate to σ' iff they have the same cycle lengths

Corollary $\#(\text{Conjugacy classes of } S_n) = P(n)$

done line

Lemma 1. Every element of A_n is a product of 3-cycles.

Pf $(12)(23) = (123)$, $(123)(234) = (12)(34) \ldots$

Lemma 2. If $N \lhd A_n$ contains a 3-cycle, then $N = A_n$

Pf WLOG, $(123) \in N$. Claim For $\sigma \in S_n$, $(123)^\sigma \in N$ $\left(\begin{matrix}\sigma \in A_n \checkmark \\ \sigma = (12)\sigma' \checkmark\end{matrix}\right)$

So N contains all 3-cycles ... $\square$

Now take $N \lhd A_n$ w/ $N \neq \{1\}$

Case 1. N contains an element w/ cycle of length ≥ 4

$\sigma = (123456)\sigma' \in N \qquad \sigma^{-1}(123)\sigma(123)^{-1} = (136)$

Case 2. N contains an element $\sigma = (123)(456)\sigma'$

consider $\sigma^{-1}(124)\sigma(124)^{-1} = (14263)$

Case 3. N contains $\sigma = (123)$ (product of pairs)

Then $\sigma^2 = (132)$ . . .

Case 4. Every element of N is a product of disjoint 2-cycles.

$\sigma = (12)(34)\sigma' \Rightarrow \sigma^{-1}(123)\sigma(123)^{-1} = (13)(24) = \tau \in N$

$\Rightarrow \tau^{-1}(125)\tau(125)^{-1} = (13452) \in N$