October 11, hours 13-14: Finish Sylow, examples, semi-direct products

October-07-11
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On board. 1. HW1 due, HW2 on web.

Theorem. 1. Sylow p-groups always exist; $Syl_p(G) \neq \emptyset$. ✓

2. Every p-group is contained in a Sylow-p group.

3. All Sylow-p subgroups of G are conjugate, and $n_p(G) := |Syl_p(G)| = 1 \bmod p$ & $n_p(G) \mid |G|$

Lemma. 1. If $P \in Syl_p(G)$ & $H < N_G(P)$ is a p-group, then $H \subset P$

2. If $P \in Syl_p(G)$, $|x| = p^\beta$, $x \in N_G(P)$, then $x \in P$.

Reformulation: $P \in Syl_p(G)$, $|H| = p^\beta \Rightarrow N_H(P) = H \cap P$

Agenda. Finish Sylow, do examples, talk about "semi-direct products.

Claim If $H \lhd HK$, $K \lhd HK$, $H \cap K = \{e\}$, then $HK \cong H \times K$.

Proof $[h,k] = hkh^{-1}k^{-1} \in H \cap K = \{e\}$ . . . . .

Corollary. If $|G| = 15$, $G = P_3 \times P_5 = \mathbb{Z}/15$.

Claim. If $(a,b) = 1$, then $\mathbb{Z}/a \times \mathbb{Z}/b \cong \mathbb{Z}/ab$

Proof. Find s, t s.t. $as + bt = 1$, and write

$$\begin{tikzcd} & \mathbb{Z}/a \arrow[dr, "\cdot b"] & \\ \mathbb{Z}/ab \arrow[ur, "\cdot t"] \arrow[dr, "\cdot s"'] & \times & \mathbb{Z}/ab \\ & \mathbb{Z}/b \arrow[ur, "\cdot a"'] & \end{tikzcd}$$

Proposition. If $P \in Syl_p(G)$, then $|\text{conjugates of } P| = 1 \bmod p$. (and $n_p \mid |G|$, of course)

Proof. P acts on the set of its conjugates by conjugation. The orbit $\{P\}$ is a singleton; by lemma, the sizes of all other orbits are divisible by p.

Proposition. If H is a p-subgroup & $P \in Syl_p(G)$, then H is contained in a conjugate of P. [In particular, all]

Proposition. If H is a p-subgroup & $P \in Syl_p(G)$, then H is contained is a conjugate of P. [In particular, all Sylow-p subgroups are conjugates]

Proof. H acts on the set of conjugates of P by conjugation. There must be a singleton orbit — a P' s.t. $H < N_G(P')$.

Semi-Direct Products. If $N < G$, $H < G$, compare $N \times H$ with NH.

There's always $\mu : N \times H \to NH$ by $(n, h) \mapsto nh$.

In general, nothing to say.

If $N \cap H = \{e\}$, injective but image might not be a group.

If $N \cap H = \{e\}$ & $N \lhd G$ & $H \lhd G$, then $[N, H] = \{e\}$ &

$$NH \cong N \times H.$$

The interesting case is when $N \cap H = \{e\}$, $N \lhd G$, H may not.

Get $H \xrightarrow{\phi} Aut(N)$ by $h \mapsto (n \mapsto n^{h^{-1}} = h n h^{-1})$

or $\phi_h(n) = h n h^{-1}$

$$n_1 h_1 n_2 h_2 = n_1 h_1 n_2 h_1^{-1} h_1 h_2 = n_1 \phi_{h_1}(n_2) h_1 h_2$$

Definition. Given abstract N, H & $\phi : H \to Aut(N)$, the semi-direct product $N \rtimes H$.

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Prop. 1. In the above case, $\mu : N \rtimes H \to NH$ is an isomorphism.

2. $N \lhd (N \rtimes H)$ and $N \rtimes H / N \cong H$.