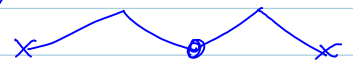
November 3, hour 24: Prime and maximal ideals

November-01-11
9:55 PM

Appeals deadline noon! No class on Tuesday!

Read Along. Selick 2.1–2.3

Riddle Along. $\mathfrak{G}($  $) = ?$

Agenda. "better ideals".

.... From now on, R is commutative.

Maximal Ideals. 1. Definition.

2. $I \subset R$ is maximal $\iff R/I$ is a field.

Example. $S = \ell^\infty = \{\text{bndd seq's in } \mathbb{R}\}$ $\quad A_n = \{(a_i) : a_n = 0\}$

Fishy

Theorem. Every ideal is contained in a maximal ideal.

Proof using Zorn's Lemma.

Theorem There exists a function

$$\mathrm{Lim} : \{\text{bndd seq's in } \mathbb{R}\} \to \mathbb{R} \quad \text{s.t.}$$

1. If (a_n) is convergent, $\lim a_n = \mathrm{Lim}\, a_n$.
2. $\mathrm{Lim}\,(a_n + b_n) = \mathrm{Lim}\,(a) + \mathrm{Lim}(b_n)$
3. $\mathrm{Lim}\,(a_n b_n) = \mathrm{Lim}(a_n) \cdot \mathrm{Lim}\,(b_n)$

\+ more....

Proof. $S = \{\text{bndd seq's in } \mathbb{R}\}$ $\quad I = \{(a_n) : a_n \neq 0 \text{ for finitely many } n\text{'s}\}$

J – a maximal ideal containing I.

$$\mathrm{Lim} : S \to S/J \underset{?}{=} \mathbb{R}$$

Prime Ideals. 1. Definition $P \subset R$ is prime if $ab \in P$
$\Rightarrow a \in P$ or $b \in P$.

2. Theorem. R/P is a domain iff P is prime.

Proof $\Rightarrow$ $ab \in P \Rightarrow [ab] = 0 \Rightarrow [a][b] = 0 \Rightarrow$ $[a]=0 \Rightarrow a \in P$ or $[b]=0 \Rightarrow b \in P$.

$\Leftarrow$ $[a][b]=0 \Rightarrow [ab]=0 \Rightarrow ab \in P \Rightarrow$ $a \in P \Rightarrow [a]=0$ or $b \in P \Rightarrow [b]=0$

Theorem. A maximal ideal is prime.

From this point, R is a Domain (commutative, no 0-divisors)

Primes. 1. $a|b$ $\quad (a|b \wedge b|a \Rightarrow a = ub)$

done before

2. $\gcd(a,b) = q$; $\gcd = q$ & $\gcd = q' \Rightarrow q' = uq$.

3. Primes: $p \neq 0$ non-unit $\quad p|ab \Rightarrow p|a$ or $p|b$

p is prime iff $\langle p \rangle$ is prime ideal.

4. Irreducible $x = ab \Rightarrow a \in R^* \vee b \in R^*$

Claim. prime $\Rightarrow$ irreducible

$p = ab \Rightarrow p|a \Rightarrow a = pc$

$\Rightarrow p = pcb \Rightarrow cb = 1 \Rightarrow b \in R^*$

Counterexample: in $\mathbb{Z}[\sqrt{-5}]$, 2 is irred (for norm reasons) but not prime, as $2 \mid (1-\sqrt{-5})(1+\sqrt{-5}) = 6$